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# THE ELEMENTS

OF

# ALGEBRA:

DESIGNED

FOR THE USE OF STUDENTS

IN THE

UNIVERSITY.

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By JOHN HIND, M.A.: F.C.P.S.: M.A.S:

LATE FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE,

CAMBRIDGE.

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SECOND EDITION.

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## ADVERTISEMENT.

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In again submitting his Work to the notice of the public, the Author deems it unnecessary to lay before the reader the reasons which induced him to attempt the performance of it, but at the same time he wishes to present him with a short account of the plan which has been adopted in its execution.

The first Chapter, as usual, consists of *Definitions* and *Introductory Remarks*, and implies a previous knowledge of the terms and fundamental Operations of Common Arithmetic. Without a competent knowledge of Arithmetic, the progress of the reader must necessarily be much retarded, and for want of it the Academical Student is in numberless instances so much discouraged, as soon to be induced to lay aside the Mathematical pursuits of the University altogether. On these grounds the acquisition of a *Facility* in Numerical Operations is particularly recommended, as, from much experience, the Author feels confident that in the end the attainment of great advantages will be secured.

In the second Chapter, the object has been to explain fully the methods of performing the *Arithmetical Operations* upon *Integral* Algebraical Quantities, or *Expressions* as they

are termed, by means of the *Signs* invented for those purposes, and they have been illustrated and exemplified by a variety of appropriate examples both original and selected; and the latter part of the Chapter points out some of the uses of Algebraical Characters and Operations in deducing Theorems respecting Magnitude in general.

The third Chapter treats of the *Greatest Common Measures* and *Least Common Multiples* of Integral Algebraical Magnitudes; and it was thought proper to introduce these subjects in their present place, because at the same time that they follow immediately from the application of the Principles and Operations previously explained, they are essentially necessary in the Reduction and Preparation of Fractions for the Arithmetical Operations upon them, which naturally follow in order.

The fourth Chapter exhibits all the Arithmetical Operations applied to Algebraical *Fractions*, and concludes with a few Deductions and Observations respecting the *Limits* of numerical Magnitude algebraically considered.

In the fifth Chapter, is developed the Treatment of *Surds* or *Irrational* Quantities, and some Observations have likewise been made respecting a Class of Algebraical Quantities, usually termed *Imaginary* or *Impossible*, which form a leading feature in some of the higher Applications of the Science.

The sixth Chapter is a short Treatise on the *Solution* of *Simple* and *Quadratic Equations*, and of such as are *reducible* to those Classes by particular Artifices or otherwise: and it

concludes with a very short Account of some of the Methods of *Elimination* or *Extermination* of unknown Quantities from Equations, and the Solution of two or more Equations subsisting in connection with each other at the same time.

The seventh Chapter gives an *Investigation* of the *Binomial Theorem,* as well as a short sketch of the *Multinomial* and *Exponential* Theorems which are deducible from it. Nothing abstruse has been attempted, and the subjects of the Chapter, as well as of each of those which precede it, have been exemplified in a variety of instances.

The subsequent Chapters of the Work comprise the *Applications* of Algebra to the Consideration of *Ratios, Proportion* and *Variation:* the common Kinds of *Progressions; Variations, Permutations* and *Combinations:* the different *Scales* of *Notation,* and a short Account of the different *Forms* and *Kinds* of Numbers, the more minute subdivisions of which may be seen in the Table of contents.

To the whole is appended a collection of miscellaneous Theorems and Problems for the practice of the student; and it is considered that the principles which have been illustrated and explained in the preceding pages of the work will be sufficient for the solution of them all.

CAMBRIDGE,
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*Lately published by the same Author,*

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THE THEORY OF EQUATIONS, &c. AND THE CALCULUS OF FINITE DIFFERENCES.

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THE

# ELEMENTS OF ALGEBRA.

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## CHAP. I.

### DEFINITIONS AND PRELIMINARY OBSERVATIONS.

ARTICLE I. DEFINITION I.

ALGEBRA is the Science which treats of the method of representing magnitudes and their relations to one another in general terms by means of symbols and signs respectively; and by such mode of representation, it comprises all particular cases of quantities and their connection with each other in general language, dependent upon the natures of the questions in which they are involved.

2. DEF. 2. The *Symbols* employed in this science are the letters of the Alphabet usually distinguished into *known* or *given*, and *unknown* or *required;* known quantities are generally denoted by the former letters $a$, $b$, $c$, &c. and unknown by the latter $x$, $y$, $z$, &c.

Sometimes the capitals $A$, $B$, $C$, &c. and the letters of the Greek alphabet $\alpha$, $\beta$, $\gamma$, &c. are adopted for the same purposes.

When a series of quantities are similarly employed in any operation, it is not unusual to represent them by the same letter with different small successive figures suffixed, as $a_0$, $a_1$, $a_2$, &c.; or by the same letter with successive numbers of accents, placed contiguous to it, as $a'$, $a''$, $a'''$, &c.

The distinction of using the former letters of the alphabet for known quantities, and the latter for unknown, though generally adopted, is not universally so, particularly in the works of the old writers.

3. Def. 3. The *Signs* here made use of are certain marks or characters invented to denote the common operations of *Addition*, *Subtraction*, *Multiplication*, *Division*, *Involution* and *Evolution*, which, by reason of the general nature and form of the quantities under consideration, can only sometimes be *effected*, but may always be *indicated* or *expressed.*

4. Def. 4. The sign of *Addition* read *plus* is $+$, and signifies that the quantity to which it is prefixed is supposed to be combined with the quantity which precedes it by the operation of addition; and all quantities to which this sign is prefixed, as well as such as have no sign expressed, are termed *positive* or *affirmative*.

Ex. 1. Thus, in the expression $a+b$, the sign $+$ indicates that the quantity represented by $b$ is to be added to that represented by $a$; and if numerical values were assigned to these quantities, this sign would be no longer necessary to *indicate* operations which could then be *effected*, for if $a$ and $b$ were 7 and 5 respectively, $a+b$ would be 12, in which the sign has disappeared.

Ex. 2. The same observations may be made respecting the expression $a+b+c+$ &c. in which any number of quantities is understood to be combined in the same manner; and it is evident that its value will be the same in whatever order the letters occur.

Ex. 3. Again, $a+a+a$ indicates that three equal quantities are to be added together, and the result of this operation, we know, would be three times any one of them, or three times $a$.

5. Def. 5. The sign of *Subtraction* called *minus* is $-$, and denotes that the quantity which it precedes is understood to be subtracted in all cases where the sign $+$ would indicate the operation of addition, and all quantities affected by it are styled *negative*.

Ex. 1. Thus, the expression $a - b$ indicates that the quantity represented by $b$ is to be subtracted from that represented by $a$, and were the quantities $a$ and $b$ numerically expressed as before, the value of $a - b$ would be $7 - 5$ or $2$.

Ex. 2. Similar remarks may be applied to such an expression as $a - b + c - d +$ &c. in which, by the last article $a$, $c$, &c. are understood to be connected by the operation of addition, and from their sum $b$, $d$, &c. are supposed to be subtracted in succession.

Ex. 3. If we take any two expressions, each made up of several quantities or terms, as $a + b - c +$ &c. and $f - g + h -$ &c. and for the sake of keeping them distinct from one another inclose each in a *Parenthesis*, or connect their parts by a line called a *Vinculum*, it follows that $(a + b - c + \&c.) - (f - g + h - \&c.)$, or $\overline{a + b - c + \&c.} - \overline{f - g + h - \&c.}$ implies that the latter of these sets of quantities is to be subtracted from the former, and it is of no consequence in what order the letters are placed, provided they all retain their proper signs.

All expressions formed by the operations indicated by the signs $+$ and $-$ are called *Compound* quantities, the parts being styled *Simple* quantities, and every quantity whatever is supposed to have one or other of these signs.

6. Def. 6. The sign of *Multiplication* read *into* is $\times$, and shews that the quantity which precedes it is to be multiplied by that which comes after it.

Ex. 1. Thus, $a \times b$ indicates the product of the quantities $a$ and $b$, and if $a$ and $b$ were numerically expressed by 24 and 6, the multiplication might be effected, and the product would be $24 \times 6$ or 144.

Ex. 2. Similarly, $a \times b \times c \times$ &c. denotes the continued product of the several quantities represented by the letters $a$, $b$, $c$, &c. each of which is called a *Factor*, and the product is said to be of as many *Dimensions* as the number of such factors contained in it.

Ex. 3. Again, $(a + b - c +$ &c.$) \times (f - g + h -$ &c.$)$ $\times (k + l - m +$ &c.$) \times$ &c. represents the continued product of the compound expressions or factors which it contains, and it is manifestly immaterial in what order they occur.

In *Algebraical* as well as in *Arithmetical* operations the place of this sign is frequently supplied by a point: thus, $a.b$ is equivalent to $a \times b$, $2.3$ to $2 \times 3$; and more generally in the multiplication of simple *Algebraical* quantities, both signs are entirely omitted, as $ab$ is supposed to be equivalent to either $a \times b$ or $a.b$: so $3 \times a$ and $5 \times b \times x$ are written $3a$ and $5bx$, and in these quantities 3 and $5b$ are styled the *Coefficients* of $a$ and $x$ respectively: hence also, the coefficient of $a$ is understood to be 1.

7. Def. 7. The sign of *Division* read *by* is $\div$ or $:$, which placed between two quantities, denotes that the former of them is intended to be divided by the latter.

Ex. 1. Thus, $a \div b$ or $a : b$, indicates that the quantity represented by $a$ is to be divided by that represented by $b$, so that, assigning to $a$ and $b$ the numerical values used in the last article, we shall have $a \div b$ equivalent to $24 \div 6$ or 4.

Ex. 2. In like manner, $(a + b - c +$ &c.$) \div (d - e + f -$ &c.$)$ denotes the result arising from the division of the former of these compound expressions by the latter.

This sign is but little used, the same operation being more generally expressed by placing the dividend over the divisor with a line between them, after the manner of a fraction: thus $\frac{a}{b}$ and $\frac{a + b - c + \&c.}{d - e + f - \&c.}$ are equivalent to the expressions used in the examples just given.

8. Def. 8. The sign of *Involution* is a small figure called an *Index* or *Exponent*, placed above the line to the right of the quantity to which it belongs, and is used merely as an abbreviation of the repetition of several multiplications.

Ex. 1. Thus, $a^2$ denotes the square of $a$, and is equivalent to $a \times a$ or $aa$, that is, to the product arising from the quantity $a$ being multiplied by itself.

Ex. 2. Similarly, $a^3$ denotes the cube of $a$, and is equivalent to $a \times a \times a$ or $aaa$: and $a^4$ represents $a \times a \times a \times a$ or $aaaa$.

Ex. 3. The same holds whatever be the number of operations, as $a^n$ denotes $a \times a \times a \times$ &c. in which the number of factors is supposed to be $n$; and hence $a^1$ will be equivalent to $a$.

Ex. 4. After the same manner, the square, cube, &c. of $a + b$ will be represented by $(a + b)^2$, $(a + b)^3$, &c.

9. Def. 9. The sign of *Evolution* called the *Radical Sign* is $\sqrt{\ }$, and denotes that the root expressed by the figure which accompanies it, is understood to be extracted from the quantity to which it is prefixed.

Ex. 1. Thus, $\sqrt[2]{a}$, which is more generally written $\sqrt{a}$, expresses the second or square root of $a$.

Ex. 2. Similarly, $\sqrt[3]{ab}$ represents the third or cube root of the quantity expressed by the product of $a$ and $b$.

Ex. 3. Again, $\sqrt[4]{a+x}$ and $\sqrt[m]{a-b}$ express respectively the fourth and $m^{\text{th}}$ roots of $a+x$ and $a-b$.

These operations are frequently expressed, as the powers were in the last article, by means of fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. $\frac{1}{m}$, placed to the right of the same quantities; thus $a^{\frac{1}{2}}$, $(ab)^{\frac{1}{3}}$, $(a+x)^{\frac{1}{4}}$ and $(a-b)^{\frac{1}{m}}$ are equivalent to the expressions used in the three examples of this article.

10. Def. 10. The *Reciprocal* of a quantity is a fraction whose numerator is unity, and denominator the quantity proposed, which consequently denotes the number of parts into which the unit is supposed to be divided.

Ex. Thus, $\frac{1}{a}$, $\frac{1}{b^2x^2}$, $\frac{1}{x^m}$, $\frac{1}{(a+x)^4}$, $\frac{1}{(ax-by)^{\frac{1}{n}}}$, &c. are the reciprocals of the quantities expressed by $a$, $b^2x^2$, $x^m$, $(a+x)^4$, $(ax-by)^{\frac{1}{n}}$, &c. which, for the sake of symmetry with the notation adopted in the last two articles, it is not unusual to write in the following forms;

$$a^{-1},\ b^{-2}x^{-2},\ x^{-m},\ (a+x)^{-4},\ (ax-by)^{-\frac{1}{n}},\ \text{\&c.}$$

11. Def. 11. The *Dimensions* of quantities are denoted by the indices or exponents which belong to them, and when the sums of such exponents are equal in all the terms of any expression, that quantity is said to be *Homogeneous*.

Ex. 1. The quantities denoted by $a$, $(b+x)^2$, $(c-y)^{\frac{1}{2}}$ and $(a-x)^{\frac{1}{m}}$ are said to be of 1, 2, $\frac{1}{2}$ and $\frac{1}{m}$ dimensions, respectively.

Ex. 2. So also, $a^{-2}$, $(a-x)^{-\frac{1}{3}}$, $(b+x)^{-m}$ and $(c+y)^{-\frac{1}{n}}$ are accounted to be of the respective dimensions

$$-2,\ -\tfrac{1}{3},\ -m \text{ and } -\frac{1}{n}.$$

12. Def. 12. Quantities are said to be *arranged* according to the dimensions of any letter involved in them, when the indices of that letter occur in the order of their magnitudes either increasing or decreasing.

Ex. 1. The terms of the expression $a^2 - ax + x^2$ are arranged according to decreasing or descending powers of $a$, and increasing or ascending powers of $x$, but it is generally said to be arranged according to the dimensions of $a$.

Ex. 2. Similarly, $x^n - ax^{n-1} + bx^{n-2} -$ &c. is arranged according to the dimensions of $x$, and $ay^2 - by^{\frac{1}{2}} + cy^{-\frac{1}{2}} +$ &c. according to the dimensions of $y$.

13. Def. 13. One algebraical expression is said to be *greater* or *less* than another according as the letter which characterises its terms is raised to higher or lower dimensions, without any regard to the particular values that might be assigned to the letters involved in it.

Ex. 1. Thus, the expression $x^3 - ax^2 + a^2x - a^3$ is said to be greater than $x^2 + ax + a^2$, though values might easily be given to $x$ and $a$ which would render the contrary to be the case.

Ex. 2. Similarly, the expression $x^m - px^{m-1} + qx^{m-2} -$ &c. is in the same sense said to be greater than

$$mx^{m-1} - (m-1)\,px^{m-2} + (m-2)\,qx^{m-3} - \text{\&c.}$$

whether $m$ be positive or negative, integral or fractional.

14. In addition to the fundamental symbols and terms of this science above explained, the sign $=$ denotes *equality*; $: :: :$ or $: = :$ expresses *proportionality* as used in common Arithmetic; $>$ is equivalent to *greater than*, $<$ to *less than*; $\because$ to *since* or *because*, and $\therefore$ to *therefore*.

Ex. 1. Thus, $ax - b = cy + d$ implies that the quantities on each side of the sign $=$ are equal to one another, and the whole expression is called an *Equation*.

Ex. 2. Also, $a : b :: c : d$ or $a : b = c : d$ denotes that $a$ has to $b$, the same ratio or relation which $c$ has to $d$, and the whole expression is styled a *Proportion*.

Ex. 3. Lastly, $\because$ $a$ is $> b$, $\therefore$ $b < a$, is read, because $a$ is greater than $b$, therefore $b$ is less than $a$.

15. All algebraical quantities are in their form either

*Integral, as* $a,\ b,\ ab,\ ab - cd,\ (ax + by)^m$, &c.

*Fractional, as* $\frac{a}{b},\ \frac{ab}{cd},\ \frac{ab+cd}{ax-by},\ \left(\frac{ax-by}{cx+ey}\right)^m$, &c.

*Irrational or Surd, as*

$$\sqrt{a},\ \sqrt[3]{\frac{ab}{c}},\ \left(\frac{ab-cd}{ax+by}\right)^{\frac{1}{4}},\ \left(\frac{ax+by}{cx-dy}\right)^{\frac{1}{m}},\ \text{\&c.}$$

*Impossible or Imaginary, as*

$$a\sqrt{-1},\ \sqrt[4]{-\frac{ab}{c}},\ \sqrt[2m]{-\frac{ab-cd}{ex+fy}},\ \text{\&c.}$$

16. In the rudiments of this science, certain additional terms are sometimes used, which in a great degree explain themselves.

Thus, instances of

*Like Quantities* are $a, 2a$; $4ab, 7ab$; $5(a+x)^m$, $11(a+x)^m$; &c.

*Unlike Quantities* are $a$, $b$; $3xy$, $5yz$; $4(a+x)^m$, $8(b+x)^n$; &c.

*Monomials* are $a$, $ab$, $abx$, &c.

*Binomials* are $a+b$, $a-bx$, $5a+7x$, &c.

*Trinomials* are $a+b+c$, $a-bx+cx^2$, $x^2-px+q$, &c.

&c. ...................................... &c.

*Multinomials* are $a+bx+cx^2+dx^3+$ &c.,

$$x^n - px^{n-1} + qx^{n-2} - \text{\&c.} + Qx^2 - Px + L, \text{ \&c.}$$

17. We may illustrate the definitions already given by the following examples, in which $a$, $b$, $c$, $d$, $e$, &c. are supposed to represent the natural numbers 1, 2, 3, 4, 5, &c.

Thus, $a+b+c-d = 1+2+3-4 = 2$:

$$ab+ac-bc+cd = 1.2+1.3-2.3+3.4 = 2+3-6+12 = 11:$$

$$(a+c)(d-b) = (1+3)(4-2) = 4.2 = 8:$$

$$\frac{a-b+c}{b+d-e} = \frac{1-2+3}{2+4-5} = \frac{2}{1} = 2:$$

$$\frac{ab+de}{ac+cd} = \frac{1.2+4.5}{1.3+3.4} = \frac{2+20}{3+12} = \frac{22}{15}:$$

$$(ac+b^2)^2 = (1.3+2^2)^2 = 7^2 = 49:$$

$$\{(a+b)(e-c)\}^3 = \{(1+2)(5-3)\}^3 = 6^3 = 216:$$

$$\left(\frac{d-a}{c-b}\right)^4 = \left(\frac{4-1}{3-2}\right)^4 = \left(\frac{3}{1}\right)^4 = 3^4 = 81:$$

$$\sqrt{abcd + a^4} = \sqrt{1.2.3.4 + 1} = \sqrt{24 + 1} = \sqrt{25} = 5:$$

$$\sqrt[3]{\frac{a^2 + bc + de}{ab + bc}} = \sqrt[3]{\frac{1 + 2.3 + 4.5}{1.2 + 2.3}} = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}.$$

In each of these instances, the *Monomials* involved are connected together by one of the signs + and – ; but it may be observed that these signs do not in any degree affect their *absolute* magnitudes, and that the terms *positive* and *negative* are applied to them merely in reference to other quantities, to which they are to be *added*, or from which they are to be *subtracted*, so that in consequence of quantity being *in general* increased by the former operation and diminished by the latter, *positive* and *negative* magnitudes are sometimes considered to be respectively *greater* and *less* than nothing.

## CHAP. II.

### *On Integral Quantities.*

---

#### I. ADDITION.

18. Def. The *Addition* of algebraical quantities is the combining or incorporating into one expression such quantities as are *like*, according to the operations indicated by their respective algebraical signs, and placing those that are *unlike* in one line, with their proper signs before them.

Ex. 1. Thus, if we have to add together the *like* positive quantities $3ax$, $4ax$ and $7ax$, it is evident that the sum will be three times the quantity $ax$, together with four times that quantity, and seven times the same quantity; that is, it will be 14 times the quantity $ax$ or $14ax$.

The same observations being equally applicable to all such quantities, we have the following

Rule 1. The sum of any number of like positive algebraical quantities is found by taking the sum of their numeral coefficients, and prefixing it to the same quantity.

Ex. 2. To add together the *like* negative quantities $-2by$, $-4by$ and $-6by$, we observe that $2by$ taken negatively, together with $4by$ and $6by$ taken also negatively, will be the same as $12by$ taken negatively, or the sum of $-2by$, $-4by$ and $-6by$ will be $-12by$, and so on: and hence, as before,

Rule 2. The sum of any number of like negative quantities is the sum of the same quantities with the negative sign placed before it.

Ex. 3. In adding together the *like* positive and negative quantities $4\,ax$, $-\,5\,ax$, $-\,2\,ax$ and $8\,ax$, we have to take $4\,ax$ and $8\,ax$ positively, that is, we must take $12\,ax$ positively: also, $5\,ax$ is to be taken negatively, and $2\,ax$ negatively, therefore, on the whole, $7\,ax$ is to be taken negatively: and hence the sum of the quantities above written will be $5\,ax$: similarly of other cases, and thence,

Rule 3. The sum of any number of like quantities with the different algebraical signs + and − is obtained by taking the excess of the sum of those with one of these signs above the sum of those with the other, and by prefixing the sign of the greater.

Ex. 4. The *unlike* quantities $3\,ax$, $-\,2\,by$, $7\,ez$, &c. having to one another no assigned numerical relation, do not admit of being incorporated into one expression, and can therefore be added together only by placing them in a line connected by their respective signs, and $3\,ax - 5\,by + 7\,ez$, &c. is called their sum: also, as the same holds of all others, we have

Rule 4. The sum of any number of unlike quantities is expressed by placing them all in a line one after another, each retaining its proper sign, which is either expressed or understood.

Ex. 5. If several quantities as $5\,ax$, $7\,bx$, $-\,8\,cx$, &c. have one or more letters common to them all, their sum will be $5\,ax + 7\,bx - 8\,cx$ &c., which according to the principle of the notation explained in (6) may manifestly be written $(5\,a + 7\,b - 8\,c \text{ &c.})\,x$ or $\overline{5\,a + 7\,b - 8\,c \text{ &c.}}\;x$: and as the same will hold in all other cases, we have

Rule 5. When one or more letters are common to several unlike quantities, their sum will be expressed by affixing the common letter or letters to the sum of the rest found by the last Rule, included in a parenthesis, or placed under a vinculum.

Ex. 6. Let it be required to find the sum of the following quantities:

$$
\begin{array}{l}
5a - 6b + 4c - 4ax + 3by - 10cz + 2ax^2 - 3by^2 + 4cz^2, \\
8a - 4b + 7c - 5ax + 8by - 15cz + 3ax^2 - 5by^2 + 7cz^2, \\
11a - 23b + 14c - 8ax + 6by - 24cz + 9ax^2 - 2by^2 + 5cz^2, \\
13a - b + 18c - 20ax + by - 3cz + 8ax^2 - 4by^2 + cz^2:
\end{array}
$$

combining each of these vertical rows of like quantities according to Rules 1 and 2, we shall have the whole sum equal to

$$37a - 34b + 43c - 37ax + 18by - 52cz + 22ax^2 - 14by^2 + 17cz^2.$$

Ex. 7. Let it be required to incorporate into one sum the following quantities:

$$
\begin{array}{l}
13x^4 + 3x^2y^2 \pm 5xy^3 + 10y^4 - ax^3 \pm b^2x^2 - c^3x + 8a^4, \\
-10x^4 - x^2y^2 \mp 8xy^3 - 5y^4 + 7ax^3 \mp 5b^2x^2 + 9c^3x - 4a^4, \\
3x^4 + 8x^2y^2 \pm 2xy^3 + 6y^4 - 6ax^3 \pm 3b^2x^2 - 2c^3x + 5a^4, \\
7x^4 - 4x^2y^2 \mp 3xy^3 - 2y^4 + 8ax^3 \mp 4b^2x^2 + 20c^3x - 18a^4;
\end{array}
$$

here, combining the quantities in each of the vertical rows according to Rule 3, and observing that the upper and lower signs of the third and sixth rows are to be taken together respectively, we shall have the sum equal to

$$13x^4 + 6x^2y^2 \mp 4xy^3 + 9y^4 + 8ax^3 \mp 5b^2x^2 + 26c^3x - 9a^4.$$

Ex. 8. Add together the following quantities:

$$
\begin{array}{l}
2ab^2 + 3ac^2 - 8cx^2 + 9b^2x - 8hy^2 - 10ky, \\
5a^3 - 4ab^2 - 7bx^2 - b^2x - 4ky^2 - 15hy, \\
14b^3 - 22ac^2 - 10x^2 + 11x - hy^2 + 5ky, \\
19ac^2 + 2ab^2 + 9x^2 - 8b^2x + 2ky^2 + 6hy;
\end{array}
$$

here observing that the quantities $ab^2$, $ac^2$ are to be taken as often positively as negatively, the sum of all such terms $= 0$, and according to Rules 3, 4 and 5, we shall have the whole sum equal to

$$5a^3 + 14b^3 - (8c + 7b + 1)x^2 + 11x - (9h + 2k)y^2 - (5k + 9h)y.$$

Ex. 9. Let it be required to find the sum of the following quantities:

$$7x^3 - 14(a+b)x + 13(a+c)y^2,$$
$$8x^2 + 5(a+b)x - 5(a+b)y^2,$$
$$-25x^2 + 8(a+b)x + 15(c-a)y^2,$$
$$17x^2 - 21(a+b)x - 10(b+d)y^2;$$

here, attending to the Rules above laid down, we have the required sum equal to

$$7x^3 - 22(a+b)x + (28c - 7a - 15b - 10d)y^2.$$

## II. SUBTRACTION.

19. Def. The *Subtraction* of algebraical quantities being the taking away of one quantity from another, is the reverse of addition, and consequently those quantities, which are to be combined with others by the operation of subtraction, must be supposed to be affected with signs contrary to what they would have been by the operation of addition.

Ex. 1. If from $4ax + 2by$ we subtract $2ax + by$, the terms of the expression $2ax + by$ must be affected with negative signs, and then the combination must be performed as in addition, so that the required difference

$$= 4ax + 2by - 2ax - by = 2ax + by.$$

Ex. 2. Taking $2az - 6ab$ from $-8az + 4ab$, we must, from the nature of the operation, add $-2az + 6ab$ to $-8az$

$+ 4\,ab$, and the sum $-10\,az + 10\,ab$, obtained on this hypothesis, is the difference of the two quantities required.

Ex. 3. In subtracting $2\,a^2 - 4\,bc$ from $6\,a^2 + 8\,bc$, it is manifest that if we take $2\,a^2$ from $6\,a^2 + 8\,bc$, we take away too much by the quantity $4\,bc$, and therefore the remainder $6\,a^2 + 8\,bc$ will be too small by this quantity: hence, therefore, adding $4\,bc$ to this remainder, the true remainder will manifestly be $4\,a^2 + 12\,bc$, which would have been obtained by changing the signs of $2\,a^2 - 4\,bc$, and proceeding as in addition.

Ex. 4. To take $6\,a^2 - 10\,ab$ from $8\,a^2 - 2\,ab$, we observe that $8\,a^2 = 6\,a^2 + 2\,a^2$, and $-2\,ab = -10\,ab + 8\,ab$, so that $8\,a^2 - 2\,ab$ is equivalent to $6\,a^2 + 2\,a^2 - 10\,ab + 8\,ab$: hence from this taking away $6\,a^2 - 10\,ab$, we have $2\,a^2 + 8\,ab$ for the remainder, and which is the same as would have been obtained, by the method pursued in the first two examples.

Similar observations being equally applicable to all other instances, we thence deduce the following general

Rule 1. Change the signs of all the quantities to be subtracted, or conceive them to be changed, and then combine them with the others by the operation of addition.

Ex. 5. If from $ax^3 - bx^2 + cx - d$, we wish to subtract $px^3 - qx^2 + rx - s$, we must take $px^3$ from $ax^3$ which gives $ax^3 - px^3 = (a-p)\,x^3 = -(p-a)\,x^3$; $-qx^2$ from $-bx^2$, which gives $qx^2 - bx^2 = (q-b)\,x^2 = -(b-q)\,x^2$; $rx$ from $cx$, which gives $cx - rx = (c-r)\,x = -(r-c)\,x$, and $-s$ from $-d$, which gives $s - d = -(d-s)$, as appears from the preceding examples: hence, therefore, the required difference will be $(a-p)\,x^3 + (q-b)\,x^2 + (c-r)\,x + (s-d)$, or $-(p-a)\,x^3 - (b-q)\,x^2 - (r-c)\,x - (d-s)$: whence we have

Rule 2. Because the signs prefixed to a parenthesis affect all the quantities included in it, the signs of the quan-

tities to be subtracted and inclosed in a parenthesis are changed or not, according as it is preceded by a negative or positive sign.

Ex. 6. From $4a^4 + 6a^3b + 8a^2b^2 + 10ab^3 + 12b^4$, let it be required to take away $2a^4 + 3a^3b + 5a^2b^2 + 8ab^3 + 11b^4$.

Conceiving the signs of the lower line to be changed, and then combining the two lines together by the rules of addition, we have the required difference

$$= 2a^4 + 3a^3b + 3a^2b^2 + 2ab^3 + b^4.$$

Ex. 7. From $6x^3y + 10x^2y^2 + 13xy^5 + 19y^4$, let $5x^3y - 2x^2y^2 + 3xy^3 - 2y^4$ be subtracted.

Here, arranging the like quantities one under another, we have

$$6x^3y + 10x^2y^2 + 13xy^5 + 19y^4,$$

$$\text{and } 5x^3y - \;\; 2x^2y^2 + \;\; 3xy^3 - \;\; 2y^4;$$

and then proceeding as before, we obtain the required remainder $= x^3y + 12x^2y^2 + 10xy^3 + 21y^4$.

Ex. 8. Let it be required to subtract from the expression $25(a^2 - x^2) - 23(ab + cx) - 17a(x + y) + 3ac$, the expression $20(a^2 + x^2) + 15(ab - cx) + 13a(x - y) + 2ac$.

First arranging the quantities as underneath, we must,

from $25(a^2 - x^2) - 23(ab + cx) - 17a(x + y) + 3ac$,

take $20(a^2 + x^2) + 15(ab - cx) + 13a(x - y) + 2ac$;

then $\because$ $25a^2 - 20a^2 = 5a^2$; $-25x^2 - 20x^2 = -45x^2$; $-23ab - 15ab = -38ab$; $-23cx + 15cx = -8cx$; $-17ax - 13ax = -30ax$; $-17ay + 13ay = -4ay$ and $3ac - 2ac = ac$; we have the required difference

$$= 5a^2 - 45x^2 - 38ab - 8cx - 30ax - 4ay + ac.$$

## III. MULTIPLICATION.

20. Def. The *Multiplication* of algebraical quantities being an abbreviated method of performing the addition or subtraction of several quantities of the same kind, will be effected according to the rules of common arithmetic, or indicated by means of the signs invented to denote that operation.

Ex. 1. If the quantity $a$ be to be multiplied by the quantity $b$, it is implied that the positive quantity $a$ is to be repeated $b$ times, and the result of such operation is written $a \times b$ or $a \, . \, b$ or $ab$, which is also positive.

Ex. 2. If the quantity $-a$ be to be multiplied by the quantity $b$, it is understood that the negative quantity $-a$ is to be taken $b$ times, and it is manifest that there will arise a negative result denoted by $-a \times b$ or $-a \, . \, b$ or $-ab$.

Ex. 3. If we have to multiply $a$ by $-b$, since from the principles of Arithmetic it is immaterial in what order quantities to be multiplied together occur, we shall have $a \times -b = -b \times a = -b \, . \, a = -ba = -ab$, which is negative.

Ex. 4. The quantity $-a$ being multiplied by the quantity $-b$ implies that $-a$ is to be subtracted $b$ times, and therefore that $a$ is to be taken positively $b$ times as appears from the last article: hence $-a \times -b$ gives a positive result $+ab$.

Ex. 5. In multiplying $a+b$ by $c+d$, we observe that $a+b$ is to be taken $c$ times positively and also $d$ times positively: that is, $(a+b)\,c$ and $(a+b)\,d$ are to be taken, or the product is $ac+bc+ad+bd = ac+ad+bc+bd$.

Ex. 6. If $a-b$ be to be multiplied by $c-d$, it is manifest that $a-b$ must be taken $c$ times positively, which gives $(a-b) \times c$ or $ac-bc$; and $d$ times negatively which gives

$(a-b)\times -d$ or $-ad+bd$: wherefore the entire result of the operation will be

$$ac-bc-ad+bd \text{ or } ac-ad-bc+bd.$$

Similar considerations in all other cases will lead to the following

Rule 1. The product of two algebraical quantities is positive or negative according as they have the same or different signs, and the product of two compound quantities is equal to the sum of the products of each of their terms.

Ex. 7. If any number of quantities $2a$, $3b$, $-4c$, &c. be to be multiplied together, we have $2a\times 3b=6ab$; $\therefore 2a\times 3b \times -4c=6ab\times -4c=-24abc$, &c.; that is, the sign of the result will manifestly be determined by the Rule above given.

Ex. 8. Let $a^2$ be to be multiplied by $a^3$; then we observe from Def. (8), that $a^2=a\times a$ and $a^3=a\times a\times a$; whence we have the product $=a^2\times a^3=(a\times a)\times(a\times a\times a)=a\times a \times a\times a\times a=a^5$, by the same definition.

Ex. 9. Again, to multiply $a^m$ by $a^n$, we must first observe that agreeably to the last mentioned definition,

$$a^m=a\times a\times a\times \text{\&c. to } m \text{ factors,}$$

$$\text{and } a^n=a\times a\times a\times \text{\&c. to } n \text{ factors;}$$

$\therefore$ the product $a^m\times a^n$

$=a\times a\times a\times$ &c. to $m$ factors $\times a\times a\times a\times$ &c. to $n$ factors

$=a\times a\times a\times$ &c. to $m+n$ factors

$=a^{m+n}$, by definition (8).

Similarly, $a^m\times a^n\times a^p\times$ &c. $=a^{m+n+p+\&c.}$; for we have just seen that $a^m\times a^n=a^{m+n}$, $\therefore$ it follows that

$$a^m\times a^n\times a^p=a^{m+n}\times a^p=a^{m+n+p}, \text{ and so on.}$$

Hence by extending this reasoning we have the following

RULE 2. The product of any number of powers of the same quantity is equal to that quantity raised to a power denoted by the sum of their indices.

Ex. 10. To multiply $x^3+ax^2+a^2x+a^3$ by $x+a$, we have $(x^3+ax^2+a^2x+a^3)\times x = x^3\times x+ax^2\times x+a^2x\times x+a^3\times x = x^4+ax^3+a^2x^2+a^3x$ and $(x^3+ax^2+a^2x+a^3)\times a = ax^3+a^2x^2+a^3x+a^4$: hence, arranging the quantities according to the dimensions of $x$, and taking their sum, we find

$$\begin{array}{rrrrr} x^4 + & ax^3 + & a^2x^2 + & a^3x & \\ & ax^3 + & a^2x^2 + & a^3x & + a^4 \\ \hline \end{array}$$

$$\therefore \text{ the product} = x^4+2ax^3+2a^2x^2+2a^3x+a^4.$$

Ex. 11. If it be required to find the product of $3x^2-2xy-y^2$ and $2x-4y$, we have as in the last example to multiply each term of $3x^2-2xy-y^2$ by each term of $2x-4y$, which operation may be exhibited as under:

$$\begin{array}{rrrr} 3x^2 - & 2xy & -y^2 & \\ 2x - & 4y & & \\ \hline 6x^3 - & 4x^2y & -2xy^2 & \\ & -12x^2y & +8xy^2 & +4y^3 \\ \hline \end{array}$$

$$\therefore \text{ the product} = 6x^3-16x^2y+6xy^2+4y^3.$$

Ex. 12. Multiply $x^3-px^2+qx-r$

by $x-a$

$$\begin{array}{rrrrr} x^4 - px^3 & + qx^2 & -rx & & \\ -ax^3 & + pax^2 & -qax & +ra & \\ \hline \end{array}$$

$\therefore$ the product

$$= x^4-(p+a)x^3+(q+pa)x^2-(r+qa)x+ra.$$

Ex. 13. Multiply $x^4-(n-1)a^2x^2+a^4$
by $x^2-a^2$

$$\begin{array}{l} x^6-(n-1)a^2x^4+a^4x^2 \\ \qquad\qquad -a^2x^4+(n-1)a^4x^2-a^6 \end{array}$$

$\therefore$ the product $=x^6-na^2x^4+na^4x^2-a^6$.

Ex. 14. Multiply $x^{3n}-px^{2n}+qx^n-r$
by $x^{3n}+px^{2n}+qx^n+r$

$$\begin{array}{l} x^{6n}-px^{5n}+qx^{4n}-rx^{3n} \\ \quad px^{5n}-p^2x^{4n}+pqx^{3n}-prx^{2n} \\ \qquad\quad qx^{4n}-pqx^{3n}+q^2x^{2n}-qrx^n \\ \qquad\qquad\quad rx^{3n}-prx^{2n}+qrx^n-r^2 \end{array}$$

$\therefore$ the product $=x^{6n}-(p^2-2q)x^{4n}+(q^2-2pr)x^{2n}-r^2$.

Ex. 15. Let it be required to find the continued product of the three compound expressions, $a^2+2ab+b^2$, $a^2-2ab+b^2$ and $a^4+2a^2b^2+b^4$.

First, multiply $a^2+2ab+b^2$
by $a^2-2ab+b^2$

$$\begin{array}{l} a^4+2a^3b+a^2b^2 \\ \quad -2a^3b-4a^2b^2-2ab^3 \\ \qquad\qquad a^2b^2+2ab^3+b^4 \end{array}$$

then, multiply $a^4-2a^2b^2+b^4$
by $a^4+2a^2b^2+b^4$

$$\begin{array}{l} a^8-2a^6b^2+a^4b^4 \\ \quad 2a^6b^2-4a^4b^4+2a^2b^6 \\ \qquad\qquad a^4b^4-2a^2b^6+b^8 \end{array}$$

$\therefore$ the continued product $=a^8-2a^4b^4+b^8$.

## IV. DIVISION.

21. Def. The *Division* of one algebraical quantity by another is the finding what quantity multiplied by the latter will produce the former, and it is therefore the reverse of multiplication, and will consequently be effected by retracing the steps, or indicated by means of the notation adopted in definition (7).

Ex. 1. Dividing $ab$ by $b$, we shall manifestly have $a$ for the quotient, because $b$ multiplied by $a$ gives $ba$ or $ab$.

Ex. 2. The quotient arising from the division of $ab$ by $-b$ is $-a$, a negative quantity, for the same reason.

Ex. 3. Similarly, the result of the division of $-ab$ by $b$ is $-a$, which is also negative.

Ex. 4. And the result of the division of $-ab$ by $-b$ is $a$, which is positive.

Ex. 5. Since $(a-b)\times(c-d)=ac-ad-bc+bd$, it follows that the quotient of $ac-ad-bc+bd$ divided by $a-b$ is $c-d$, which may be obtained by the process underneath:

$$
\begin{array}{rl}
a-b) & ac-ad-bc+bd\ (c-d, \\
 & \underline{ac-bc\qquad\qquad} \\
 & \qquad -ad+bd \\
 & \qquad \underline{-ad+bd}
\end{array}
$$

for it is manifest that $ac$ divided by $a$ gives $c$, and if the whole of the divisor be multiplied by this quantity and the product be subtracted, there remains $-ad+bd$: the same operation being repeated, the other term $-d$ of the quotient is obtained.

By a similar mode of reasoning in all other cases, we shall obtain

Rule 1. The division of one algebraical quantity by another must be performed as in common arithmetic, and the quotient will be positive or negative, according as the divisor and dividend are affected with the same or different signs.

Ex. 6. To divide $a^4$ by $a^2$, we must refer to definition (8) where we find that $a^4$ is equivalent to $a \times a \times a \times a$ and $a^2$ to $a \times a$: wherefore, dividing the former of these by the latter and observing that the multiplication by two factors is neutralized by the division by the same two, we get the quotient $= a \times a$ or $a^2$.

Ex. 7. Again, since $a^m$ is equivalent to $a \times a \times a \times$ &c. to $m$ factors and $a^n$ to $a \times a \times a \times$ &c. to $n$ factors, it follows that, when $m$ is greater than $n$, the quotient of $a^m$ divided by $a^n$ is equivalent to $a \times a \times a \times$ &c. to $m-n$ factors; that is, $a^m \div a^n$ is equal to $a^{m-n}$.

From such instances as these we obtain the following

Rule 2. The division of one power of a quantity by another power of the same quantity is effected by subtracting the index or exponent of the divisor from that of the dividend.

Ex. 8. Since $a^m \div a^n = a^{m-n}$, if we suppose $m = n$, we obviously have $1 = a^{m-m} = a^0$; whence it follows from the notation adopted in (10), that any quantity whatever raised to the power denoted by 0 is an expression equivalent to 1.

Ex. 9. To divide $9a^2bc - 12ab^2c + 15abc^2$ by $3ab$, we observe that $9a^2bc \div 3ab = 3ac$; $-12ab^2c \div 3ab = -4bc$, and $15abc^2 \div 3ab = 5c^2$; whence the entire quotient will be $3ac - 4bc + 5c^2$.

Ex. 10. To divide $a^2+6ab+8b^2$ by $a+4b$, we must arrange the quantities according to the dimensions of one of the letters involved, and proceed as in common arithmetic: thus, the arrangement being already according to the dimensions of $a$, we have

$$\begin{array}{l} a+4b)\ a^2+6ab+8b^2\ (a+2b, \\ \qquad\qquad a^2+4ab \\ \hline \qquad\qquad\quad 2ab+8b^2 \\ \qquad\qquad\quad 2ab+8b^2 \\ \hline \end{array}$$

in which we first enquire what is the result of the division of $a^2$ by $a$, and this being $a$ is placed in the quotient: the divisor is then multiplied by $a$, and the product subtracted from the dividend leaves a remainder $2ab+8b^2$: similarly the quotient of $2ab$ by $a$ is $2b$, and the process being continued, the remainder becomes 0, and thus the division is completed.

Ex. 11. To divide $a^4+4a^2b^2+16b^4$ by $a^2-2ab+4b^2$, we arrange the quantities as underneath:

$$\begin{array}{l} a^2-2ab+4b^2)\ a^4+4a^2b^2+16b^4\ (a^2+2ab+4b^2, \\ \qquad\qquad\qquad a^4-2a^3b+4a^2b^2 \\ \hline \qquad\qquad\qquad\quad 2a^3b+16b^4 \\ \qquad\qquad\qquad\quad 2a^3b-4a^2b^2+8ab^3 \\ \hline \qquad\qquad\qquad\qquad\qquad 4a^2b^2-8ab^3+16b^4 \\ \qquad\qquad\qquad\qquad\qquad 4a^2b^2-8ab^3+16b^4 \\ \hline \end{array}$$

where the steps of the operation are effected in the same manner as in the last example, and the remainder is 0.

Ex. 12. To divide $a^m-x^m$ by $a-x$, we have the operation as under:

$$a-x)\ a^m - x^m\ (a^{m-1} + a^{m-2}x + a^{m-3}x^2 + \&c. + ax^{m-2} + x^{m-1},$$
$$\frac{a^m - a^{m-1}x}{\quad}$$
$$a^{m-1}x - x^m$$
$$\frac{a^{m-1}x - a^{m-2}x^2}{\quad}$$
$$a^{m-2}x^2 - x^m$$
$$\frac{a^{m-2}x^2 - a^{m-3}x^3}{\quad}$$
$$a^{m-3}x^3 - x^m, \text{ and so on:}$$

now, it is observable that in the remainder the index of $a$ is diminished and that of $x$ increased by unity in each step, and that the sum of the indices in every term always $= m$; whence we shall at length have $a^2x^{m-2} - x^m$ for a remainder: and continuing the division from this as before, we obtain

$$a^2x^{m-2} - x^m$$
$$\frac{a^2x^{m-2} - ax^{m-1}}{\quad}$$
$$ax^{m-1} - x^m$$
$$\frac{ax^{m-1} - x^m}{\quad}$$

whence it follows, that if $m$ be any positive whole number whatever, $a^m - x^m$ is divisible by $a - x$ without a remainder, and the division gives the following $m$ terms

$$a^{m-1} + a^{m-2}x + \&c. + ax^{m-2} + x^{m-1} \text{ for the quotient.}$$

Ex. 13. Let $x^3 - px^2 + qx - r$ be to be divided by $x - a$; then as before, we have

$$x-a)\ x^3 - px^2 + qx - r\ (x^2 + (a-p)\,x + (a^2 - pa + q),$$
$$\frac{x^3 - ax^2}{\quad}$$
$$+(a-p)\,x^2 + qx$$
$$\frac{+(a-p)\,x^2 - (a^2 - pa)\,x}{\quad}$$
$$(a^2 - pa + q)\,x - r$$
$$\frac{(a^2 - pa + q)\,x - (a^3 - pa^2 + qa)}{\quad}$$
$$a^3 - pa^2 + qa - r,$$

the steps of the operation being effected as in the preceding examples, the remainder is $a^3 - pa^2 + qa - r$, which, it may be observed, is the same as the dividend with $a$ in the place of $x$.

Ex. 14. If we divide 1 by $1 + x$, we shall have the following operation:

$$
\begin{array}{l}
1 + x)\ 1 \qquad (1 - x + x^2 - x^3 + \&c. \\
\qquad\ \ \underline{1 + x} \\
\qquad\quad - x \\
\qquad\quad \underline{- x - x^2} \\
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad \underline{x^2 + x^3} \\
\qquad\qquad\qquad - x^3 \\
\qquad\qquad\qquad \underline{- x^3 - x^4} \\
\qquad\qquad\qquad\qquad x^4,
\end{array}
$$

wherein we observe that the index of $x$ in the remainder is always equal to the corresponding number of terms in the quotient: and it is manifest that by continuing the operation, the number of terms might be *indefinitely* increased, and form what is called an *Infinite Series*.

## V. INVOLUTION.

22. Def. The *Involution* of algebraical quantities being the repetition of one or more multiplications will be effected by means of the rules already given for that operation, or indicated according to the notation explained in definition (8).

Ex. 1. The square of $ab = ab \times ab = a^2b^2$; the cube of $ab = ab \times ab \times ab = a^3b^3$; the fourth power of $ab = ab \times ab \times ab \times ab = a^4b^4$; &c.

Ex. 2. The square of $-xy=(-xy)\ (-xy)=x^2y^2$; the cube of $-xy=(-xy)\ (-xy)\ (-xy)=-x^3y^3$; the fourth power of $-xy=(-xy)\ (-xy)\ (-xy)\ (-xy)=x^4y^4$; &c.

Ex. 3. The square of $a^2=a^2\times a^2=a^{2+2}=a^4$; the cube of $a^2=a^2\times a^2\times a^2=a^{2+2+2}=a^6$; the fourth power of $a^2=a^2\times a^2\times a^2\times a^2=a^{2+2+2+2}=a^8$; and so on.

Ex. 4. The $n^{\text{th}}$ power of $x^m=x^m\times x^m\times x^m\times$ &c. to $n$ factors $=x^{m+m+m+\text{ \&c. to } n \text{ terms}}=x^{mn}$; the $n^{\text{th}}$ power of $-x^m$

$$=(-x^m)\times(-x^m)\times(-x^m)\times\text{ \&c. to } n \text{ factors}=\pm\, x^{mn},$$

where the positive or negative sign is to be used according as $n$ is an even or an odd number.

Ex. 5. The $n^{\text{th}}$ power of $x^2y=x^2y\times x^2y\times x^2y\times$ &c. to $n$ factors $=x^2\times x^2\times x^2\times$ &c. to $n$ factors $\times\, y\times y\times y\times$ &c. to $n$ factors

$$=x^{2+2+2+\text{ \&c. to } n \text{ terms}}\times y^{1+1+1+\text{ \&c. to } n \text{ terms}}=x^{2n}\times y^n=x^{2n}y^n.$$

From what has been proved above, we derive

Rule 1. A simple algebraical quantity is raised to any power by multiplying the index of each factor by that of the proposed power: and the result will always be positive when the index of that power is even, and will have the same sign as the quantity itself when that index is odd.

Ex. 6. The square, cube, fourth power, &c. of $a+x$ will be indicated by $(a+x)^2$, $(a+x)^3$, $(a+x)^4$, &c. but these operations will be effected as under, by actual multiplication, thus:

$$\begin{array}{rl}
\text{the root} = & a + x \\
& a + x \\
\hline
& a^2 + ax \\
& \qquad ax + x^2 \\
\hline
\therefore\ \text{the square} = & a^2 + 2ax + x^2 \\
& a + x \\
\hline
\end{array}$$

$$
\begin{array}{rl}
& a^3 + 2\,a^2x + \phantom{3}\,ax^2 \\
& \phantom{a^3 + {}} a^2x + 2\,ax^2 + \phantom{3}\,x^3 \\
\hline
\therefore \text{ the cube} = & a^3 + 3\,a^2x + 3\,ax^2 + \phantom{3}\,x^3 \\
& a + x \\
\hline
& a^4 + 3\,a^3x + 3\,a^2x^2 + \phantom{3}\,ax^3 \\
& \phantom{a^4 + {}} a^3x + 3\,a^2x^2 + 3\,ax^3 + x^4 \\
\hline
\end{array}
$$

$\therefore$ the fourth power $= a^4 + 4\,a^3x + 6\,a^2x^2 + 4\,ax^3 + x^4$,

and so on; and since each repetition of the multiplication increases the numerical coefficient of the second term by unity, it follows that the coefficient of the second term is always equal to the index: and it may be further observed that the indices of $a$ and $x$ descend and ascend respectively by unity in each succeeding term, and hence we shall have

$$(a + x)^m = a^m + m\,a^{m-1}x + \&c.$$

Whence we immediately deduce

Rule 2. The involution of a compound algebraical quantity may either be indicated by the proper exponent, or effected by actual multiplication.

Ex. 7. The square of $a = a^2$; the square of $(a + 1)$ $= (a + 1) \times (a + 1) = a^2 + 2\,a + 1$, from which it appears that if the root be increased by 1, the square will be increased by twice the original root + 1.

Ex. 8. The cube of $a = a^3$; the cube of $a + 1 = (a + 1)$ $\times\ (a + 1) \times (a + 1) = a^3 + 3\,a^2 + 3\,a + 1$: whence it is manifest that if the root be increased by 1, the cube will be increased by three times the square of the original root + three times the original root + 1. Similarly of higher powers.

Ex. 9. To find the successive powers of $a+b+c$, expressed by $(a+b+c)^2$, $(a+b+c)^3$, &c. we proceed by multiplication as underneath:

$$\begin{array}{lllll}
\text{the root} = & a+b+c & & & \\
 & a+b+c & & & \\
\hline
 & a^2+ab+ac & & & \\
 & \quad\ \ ab+b^2+bc & & & \\
 & \qquad\quad\ ac+bc+c^2 & & & \\
\hline
\therefore \text{ the square} = & a^2+b^2+c^2+2(ab+ac+bc) & & & \\
 & a\ +b\ +c & & & \\
\hline
 & a^3\ +ab^2+ac^2+2(a^2b+a^2c+abc) & & & \\
 & a^2b+\ b^3+bc^2+2(ab^2+abc+b^2c) & & & \\
 & a^2c+b^2c+\ c^3+2(abc+ac^2+bc^2) & & & \\
\hline
\end{array}$$

$\therefore$ the cube

$$= a^3+b^3+c^3+3(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+6abc,$$

and so on.

The same results might however have been obtained by considering $b+c$ as one quantity, thus

$$(a+b+c)^2 = \{a+(b+c)\}^2 = a^2+2a(b+c)+(b+c)^2$$

$$= a^2+2ab+2ac+b^2+2bc+c^2 = a^2+b^2+c^2+2(ab+ac+bc):$$

$$(a+b+c)^3 = \{a+(b+c)\}^3 = a^3+3a^2(b+c)+3a(b+c)^2$$

$$+(b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)$$

$+6abc$: and a similar process may be adopted for higher indices and more terms.

Ex. 10. To find the square, cube, &c. of $2a-b+3x+y$, we may divide it into two parts, thus $(2a-b)+(3x+y)$, and then we shall have

$$
\begin{array}{rl}
\text{the root} = & (2a-b)+(3x+y) \\
 & (2a-b)+(3x+y) \\
\hline
 & (2a-b)^2+(2a-b)\,(3x+y) \\
 & \qquad +(2a-b)\,(3x+y)+(3x+y)^2 \\
\hline
\end{array}
$$

$$\therefore \text{ the square} = (2a-b)^2+2\,(2a-b)\,(3x+y)+(3x+y)^2$$
$$=4a^2-4ab+b^2+12ax-6bx+4ay-2by+9x^2+6xy+y^2:$$

and similarly of the cube, fourth, &c. powers; and we may remark that it is of no consequence which of the quantities are taken together, and that this method may be extended to any number of quantities whatever.

## VI. EVOLUTION.

23. Def. The *Evolution* of algebraical quantities is the reverse of involution, and will therefore be effected by retracing the steps of that operation, or indicated according to the notation pointed out in definition (9) of the first chapter.

Ex. 1. The square root of $a^2$ is $\pm a$, because the square of either of the quantities $+a$ and $-a$ is $a^2$.

Ex. 2. The cube root of $-x^3y^6$ is $-xy^2$, since, by the last article, we have $(-xy^2)^3 = -x^3y^6$.

Ex. 3. The $m^{\text{th}}$ root of $x^m$ is $x$, because $x$ raised to the $m^{\text{th}}$ power is $x \times x \times x \times$ &c. to $m$ factors $= x^m$.

Ex. 4. The $m^{\text{th}}$ root of $x^{mn}$ is $x^n$, because $x^{mn} = x^m \times x^m \times x^m \times$ &c. to $n$ factors, and thence the $m^{\text{th}}$ root of $x^{mn} = x \times x \times x \times$ &c. to $n$ factors $= x^n$. Whence we have

Rule 1. The operation of evolution is effected by dividing the index or indices of the proposed quantity by the number

denoting the root to be extracted, and the root will have the same sign as the quantity when this number is odd, and both signs when it is even.

Ex. 5. Since the square of $a+b$ is $a^2+2ab+b^2$, in order to obtain the square root of $a^2+2ab+b^2$, we must consider by what process the quantity $a+b$ can be generally derived from it.

Now, in the first place, it is readily observed that $a$ the first term of the root, is the square root of $a^2$ the first term of the square; and in addition to this there still remains $2ab+b^2$ from which $b$ is to be obtained: but $2ab+b^2 = (2a+b)b$, and therefore $b$ will be determined by dividing the first term of the remainder by twice the first term of the root, and to complete the operation twice this first term together with the second must be multiplied by the second, and after subtraction there is no remainder.

Ex. 6. Because $(a+b-c)^2 = a^2+2ab+b^2-2ac-2bc+c^2$, we shall have the square root of this latter quantity $=a+b-c$, which may be obtained from it according to the method pointed out in the last example, and the operation will stand as underneath:

$$
\begin{array}{rl}
 & a^2+2ab+b^2-2ac-2bc+c^2\ (a+b-c, \\
 & a^2 \\
\hline
2a+b) & 2ab+b^2 \\
 & 2ab+b^2 \\
\hline
2a+2b-c) & -2ac-2bc+c^2 \\
 & -2ac-2bc+c^2 \\
\hline
\end{array}
$$

wherein the operation as above explained being repeated leads to both the second and third terms of the root.

The same method extends to all other cases, and thence to extract the square root of a compound quantity, we have the following general

RULE 2. Arrange the terms in the order of the magnitudes of the indices of some one quantity, or in the order of its dimensions: find the square root of the first term, and subtract its square from the proposed quantity: bring down the next two terms and find the next term of the root by dividing this last quantity by twice the first, and affix it with its proper sign to the divisor: multiply this result by the said second term of the root: bring down to the remainder as many terms as may make the number equal to that in the next completed divisor: and thus continue the process till the root, or the requisite approximation to it, is obtained.

Ex. 7. Required the square root of

$$4a^4 - 12a^3 + 21a^2 - 18a + 9.$$

Here the terms are already arranged so that the indices of the quantity $a$ descend regularly, and therefore by the immediate application of the last rule, we have the operation as under:

$$
\begin{array}{rl}
 & 4a^4 - 12a^3 + 21a^2 - 18a + 9\ (2a^2 - 3a + 3. \\
 & 4a^4 \\
\hline
4a^2 - 3a) & -12a^3 + 21a^2 \\
 & -12a^3 + \ \ 9a^2 \\
\hline
4a^2 - 6a + 3) & 12a^2 - 18a + 9 \\
 & 12a^2 - 18a + 9 \\
\hline
\end{array}
$$

Ex. 8. Find the square root of

$$16 - 48x + 44x^2 - 12x^3 + x^4.$$

In this case the terms are arranged according to ascending powers of $x$, and thence, as before, we have

$$
\begin{array}{rl}
& 16 - 48x + 44x^2 - 12x^3 + x^4 \,(4 - 6x + x^2. \\
& 16 \\
\hline
8 - 6x) & -48x + 44x^2 \\
& -48x + 36x^2 \\
\hline
8 - 12x + x^2) & 8x^2 - 12x^3 + x^4 \\
& 8x^2 - 12x^3 + x^4 \\
\hline
\end{array}
$$

Ex. 9. Find the square root of

$$x^3(x - 2a) + a^2b(b - 2x) + (a^2 + 2ab)x^2.$$

The arrangement required being here made, we have to apply the rule to $x^4 - 2ax^3 + (a^2 + 2ab)x^2 - 2a^2bx + a^2b^2$, or to $a^2b^2 - 2a^2bx + (a^2 + 2ab)x^2 - 2ax^3 + x^4$, as it is a matter of indifference whether we suppose the indices of $x$ to ascend or descend: and taking the former supposition we have the following operation:

$$
\begin{array}{rl}
& x^4 - 2ax^3 + (a^2 + 2ab)x^2 - 2a^2bx + a^2b^2 \,(x^2 - ax + ab. \\
& x^4 \\
\hline
2x^2 - ax) & -2ax^3 + a^2x^2 \\
& -2ax^3 + a^2x^2 \\
\hline
2x^2 - 2ax + ab) & 2abx^2 - 2a^2bx + a^2b^2 \\
& 2abx^2 - 2a^2bx + a^2b^2 \\
\hline
\end{array}
$$

Ex. 10. Extract the square root of $x^{2m} + 2x^my^m - y^{2m}$.

The quantities here being already properly arranged, we have the operation as follows:

$$
\begin{array}{rl}
 & x^{2m} + 2x^m y^m - y^{2m} \,(x^m + y^m, \\
 & x^{2m} \\
\hline
2x^m + y^m) & +2x^m y^m - y^{2m} \\
 & +2x^m y^m + y^{2m} \\
\hline
 & \qquad -2y^{2m},
\end{array}
$$

and in this case the remainder being $-2y^{2m}$, it appears that the proposed quantity is not an exact square, and consequently that its square root cannot be accurately obtained.

Ex. 11. Since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we shall have the cube root of the latter quantity $= a + b$; and it remains to be determined in what manner it may be deduced from it.

Now, it is obvious that the first term $a$ of the root is the cube root of $a^3$ the first term of the proposed quantity: hence, taking away $a^3$, we have $3a^2b + 3ab^2 + b^3$ left to enable us to find $b$: but $3a^2b + 3ab^2 + b^3 = b(3a^2 + 3ab + b^2)$, and thence it is manifest that $b$ will be obtained by dividing the first term of the remainder by thrice the square of $a$; and to complete the divisor we must add to it three times the product of the two terms and also the square of the last: thus the second term being obtained, the repetition of a similar process may manifestly be adopted if the quantity proposed contain more terms.

This example furnishes a rule, which might, if necessary, be enunciated at length in the same manner as that for the square root has been, and it is not difficult to perceive that a similar method may be applied to the extraction of the fourth, fifth, &c. roots of any compound quantity whatever.

Ex. 12. If it be required to find the cube root of

$$a^6 - 3(a^5 + a) + 5a^3 - 1,$$

we first arrange the terms according to the dimensions of $a$, and then the operation will stand as underneath:

$$
\begin{array}{r l}
 & a^6 - 3a^5 + 5a^3 - 3a - 1\ (a^2 - a - 1, \\
 & a^6 \\
\hline
3a^4 - 3a^3 + a^2) & -3a^5 + 5a^3 - 3a \\
 & -3a^5 + 3a^4 - a^3 \\
\hline
3a^4 - 6a^3 + 3a + 1) & -3a^4 + 6a^3 - 3a - 1 \\
 & -3a^4 + 6a^3 - 3a - 1 \\
\hline
\end{array}
$$

in which it may be observed that the complete divisors are formed as in the last example.

Ex. 13. To find the fourth root of

$$a^4x^4 - 4a^3bx^3 + 6a^2b^2x^2 - 4ab^3x + b^4.$$

Here, it is evident that the fourth root is the square root of the square root, and therefore the operation may stand as follows:

$$
\begin{array}{r l}
 & a^4x^4 - 4a^3bx^3 + 6a^2b^2x^2 - 4ab^3x + b^4\ (a^2x^2 - 2abx + b^2, \\
 & a^4x^4 \\
\hline
2a^2x^2 - 2abx) & -4a^3bx^3 + 6a^2b^2x^2 \\
 & -4a^3bx^3 + 4a^2b^2x^2 \\
\hline
2a^2x^2 - 4abx + b^2) & 2a^2b^2x^2 - 4ab^3x + b^4 \\
 & 2a^2b^2x^2 - 4ab^3x + b^4 \\
\hline
\end{array}
$$

Whence $a^2x^2 - 2abx + b^2$ is the square root of the quantity proposed: and repeating the operation, we have

$$
\begin{array}{r l}
 & a^2x^2 - 2abx + b^2\ (ax - b, \\
 & a^2x^2 \\
\hline
2ax - b) & -2abx + b^2 \\
 & -2abx + b^2 \\
\hline
\end{array}
$$

so that $ax - b$ is the fourth root required.

Ex. 14. Let it be required to find the sixth root of

$$a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$$

The sixth root being the cube root of the square root, we shall have

$$\begin{array}{rl}
 & a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6 \quad (a^3 - 3a^2x + 3ax^2 - x^3, \\
 & a^6 \\
\hline
2a^3 - 3a^2x) & -6a^5x + 15a^4x^2 \\
 & -6a^5x + 9a^4x^2 \\
\hline
2a^3 - 6a^2x + 3ax^2) & 6a^4x^2 - 20a^3x^3 + 15a^2x^4 \\
 & 6a^4x^2 - 18a^3x^3 + 9a^2x^4 \\
\hline
2a^3 - 6a^2x + 6ax^2 - x^3) & -2a^3x^3 + 6a^2x^4 - 6ax^5 + x^6 \\
 & -2a^3x^3 + 6a^2x^4 - 6ax^5 + x^6 \\
\hline
\end{array}$$

$\therefore\ a^3 - 3a^2x + 3ax^2 - x^3$ is the square root of the proposed quantity, and of this the cube root is required, therefore,

$$\begin{array}{rl}
 & a^3 - 3a^2x + 3ax^2 - x^3 \ (a - x, \\
 & a^3 \\
\hline
3a^2 - 3ax + x^2) & -3a^2x + 3ax^2 - x^3 \\
 & -3a^2x + 3ax^2 - x^3 \\
\hline
\end{array}$$

whence the sixth root required is $a - x$.

A similar process may be adopted in all cases where the number denoting the root to be extracted is even, and it will then remain to devise a method by which the odd roots may be determined.

The following method is general, and exceedingly simple in its operations.

In the sixth example of (22) we have seen that

$$(a + x)^m = a^m + ma^{m-1}x + \&c.$$

from which it is obvious that the second term of the $m^{th}$ root may be obtained by dividing the second term of the proposed quantity by $ma^{m-1}$, or by $m$ times the first term raised to the $(m-1)^{th}$ power: and if the whole of the terms in the root thus obtained be involved, and the result subtracted from the quantity proposed and the process be repeated, any root of any compound quantity whatever may be readily obtained.

Ex. 15. To extract the cube root of $a^3 - 3a^2b + 3a^2 + 3ab^2 + 3a - b^3 + 3b^2 - 3b + 1$, we have

$$a^3 - 3a^2b + 3a^2 + 3ab^2 + 3a - b^3 + 3b^2 - 3b + 1\ (a - b + 1.$$
$$a^3$$
$$\overline{\phantom{xxxxxxxx}}$$
$$3 \times a^2 = 3a^2) - 3a^2b$$
$$\therefore (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$\overline{\phantom{xxxxxxxx}}$$
$$3a^2) + 3a^2$$
$$\therefore (a - b + 1)^3 = a^3 - 3a^2b + 3a^2 + 3ab^2 + 3a - b^3 + 3b^2 - 3b + 1$$

Ex. 16. To extract the fifth root of $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$, we proceed in a similar way, thus:

$$x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5\ (x - 2y.$$
$$x^5$$
$$\overline{\phantom{xxxxxxxx}}$$
$$5 \times x^4 = 5x^4) - 10x^4y$$
$$\therefore (x - 2y)^5 = x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5.$$

## MISCELLANEOUS OBSERVATIONS AND DEDUCTIONS.

24. Having, in the preceding pages, endeavoured to shew how all the fundamental operations of Arithmetic may be applied to Algebraical quantities, and pointed out how these

operations may always be *indicated*, and how they may generally be *effected*, we shall, in the remainder of the present chapter, consider their application in various instances, some of which involve propositions of the utmost importance to the student's progress in the more advanced parts of the science: and it will readily be perceived how the general relations of abstract quantities to one another may be discovered by means of the letters and symbols explained in the introductory chapter, and of the operations, the methods of performing which have been laid down and exemplified in this.

25. Precisely as in algebraical quantities may the fundamental operations explained in the preceding pages be applied to numerical magnitudes: thus, we may find the sum and difference of 584 and 326 by considering that

$$584 = 500 + 80 \;\; + 4$$

$$326 = 300 + 20 \;\; + 6$$

$$\therefore \text{ the sum } = 800 + 100 + 10 = 910:$$

$$\text{and the difference } = 200 + 60 \;\; - 2 \;\; = 258.$$

Again, to find the product of 351 and 26, we may put these numbers in the following forms:

$$351 = 300 \;\; + 50 \;\;\; + 1$$

$$\text{and} \quad 26 = 20 \;\;\; + 6$$

$$6000 + 1000 + 20$$

$$1800 + 300 + 6$$

$$\therefore \text{ the product } = 6000 + 2800 + 320 + 6 = 9126.$$

Whence to divide 9126 by 26, we observe that the dividend and divisor are equivalent to $6000 + 2800 + 320 + 6$ and $20 + 6$ respectively, and the operation will stand as follows:

$$
\begin{array}{rl}
20+6)\ 6000+2800+320+6\ (300+50+1, & \\
6000+1800 & \\
\hline
1000+320 & \\
1000+300 & \\
\hline
20+6 & \\
20+6 & \\
\hline
\end{array}
$$

so that the quotient is 351 as it manifestly ought.

The Involution and Evolution of numerical magnitudes may be effected by means of similar considerations, but the importance of these operations in Arithmetic will entitle them to a more minute discussion.

26. To find the successive powers of any number as 29, which is equivalent to $20+9$, we may proceed by effecting the successive multiplications as in the preceding pages: thus,

$$
\begin{array}{rl}
\text{the root} = & 20+9 \\
& 20+9 \\
\hline
& 400+180 \\
& \quad\quad 180+81 \\
\hline
\therefore \text{ the square} = & 400+360+81 = 841; \\
& 20+9 \\
\hline
& 8000+7200\ +1620 \\
& \quad\quad 3600\ +3240+729 \\
\hline
\end{array}
$$

$$\therefore \text{ the cube} = 8000+10800+4860+729 = 24389:$$

and so on for succeeding powers; and the same principle may manifestly be applied to numbers consisting of any number of figures whatever.

27. In example (7) of article (22), it has been observed that if the root be increased by 1, the square will be increased by twice the original root $+1$, and this may readily be exemplified numerically: thus,

$$\begin{array}{r} 19 = 18 + 1 \\ 18 + 1 \\ \hline 324 + 18 \qquad \\ 18 + 1 \\ \hline \end{array}$$

$\therefore$ the square of $19 = 324 + 36 + 1 = 361$:

which circumstance will also enable us easily to deduce the square of any number from that which immediately precedes it: thus

the square of $51 = 50^2 + 2.50 + 1 = 2500 + 100 + 1 = 2601$, &c.

Upon the same principle, if the root be diminished by 1; the square will be diminished by twice the original root $-1$: and thus we shall have

$$\begin{array}{r} 24 = 25 - 1 \\ 25 - 1 \\ \hline 625 - 25 \qquad \\ -25 + 1 \\ \hline \end{array}$$

$\therefore$ the square of $24 = 625 - 50 + 1 = 625 - (50 - 1) = 576$,

as may easily be proved to be correct.

From this consideration we are enabled to deduce with facility the square of any number from that which immediately succeeds it: thus,

the square of $49 = 50^2 - 2.50 + 1 = 2500 - 100 + 1 = 2401$.

28. In the same manner from the result of example (8), of article (22), the cube of any number may be derived from the cube of that which immediately precedes it: thus,

$$\text{the cube of } 21 = 20^3 + 3.20^2 + 3.20 + 1$$

$$= 8000 + 1200 + 60 + 1 = 9261:$$

and by an extension of the rule analogous to that used in the latter part of the last article, we have

$$\text{the cube of } 19 = 20^3 - 3.20^2 + 3.20 - 1$$

$$= 8000 - 1200 + 60 - 1 = 6859.$$

Similar processes may be adopted for higher powers, and will in many cases greatly facilitate arithmetical operations.

29. From the second rule of article (23), may be readily deduced the square root of any numerical magnitude: thus since $441 = 400 + 40 + 1$, to obtain its square root we have the following operation:

$$
\begin{array}{rl}
 & 400 + 40 + 1 \ (20 + 1, \\
 & 400 \\
\hline
40 + 1) & 40 + 1 \\
 & 40 + 1 \\
\hline
\end{array}
$$

and $\therefore$ the square root of $441 = 20 + 1 = 21$.

Again, to find the square root of 5184, we observe that it may be put in the form $4900 + 280 + 4$, so that the operation may stand as follows:

$$
\begin{array}{rl}
 & 4900 + 280 + 4 \ (70 + 2, \\
 & 4900 \\
\hline
140 + 2) & 280 + 4 \\
 & 280 + 4 \\
\hline
\end{array}
$$

and therefore the root required is 72.

30. The same method might be resorted to in all other instances, but it is obvious that some dexterity would be required to separate the numbers properly into their component parts, and this will be superseded by the following consideration.

$$\begin{aligned} \text{Since the square of } 10 &= 100, \\ \text{the square of } 100 &= 10000, \\ \text{the square of } 1000 &= 1000000, \\ \&\text{c.} \ldots\ldots\ldots &= \&\text{c.} \end{aligned}$$

it will manifestly follow that the square roots of numbers of less than three, five, seven, &c. figures must consist of one, two, three, &c. figures respectively, so that if a point be placed over each alternate figure beginning at the unit's place, the number of points over the quantity proposed will denote the number of figures of which the root consists.

Ex. 1. To extract the square root of 273529.

$$\begin{array}{r l} & 2\dot{7}3\dot{5}2\dot{9}\ (523 = \text{the square root;} \\ & 25 \\ \hline 102) & 235 \\ & 204 \\ \hline 1043) & 3129 \\ & 3129 \\ \hline \end{array}$$

here the points being properly placed and the square next less than 27 being 25, the first term in the root is 5: to the remainder 2 the next two figures are then annexed and the quotient 10 is formed by doubling the first figure 5 of the root: in 23 the divisor 10 is contained twice, so that the second figure of the root is 2, which is annexed to the divisor and the multiplication is then effected: and a repetition of the same process determines the remaining figure 3 of the root required.

If we supply the cyphers understood in the operation, the same thing may be exhibited as follows:

$$
\begin{array}{rr}
 & 2\dot{7}3\dot{5}2\dot{9}\ (500 + 20 + 3 = 523. \\
500^2 = & 250000 \\
\hline
1000 + 20 = 1020) & 23529 \\
 & 20400 \\
\hline
1040 + 3 \;=\; 1043) & 3129 \\
 & 3129 \\
\hline
\end{array}
$$

Ex. 2. To extract the square root of 190968.

$$
\begin{array}{rl}
 & 1\dot{9}0\dot{9}6\dot{8}\ (436, \\
 & 16 \\
\hline
83) & 309 \\
 & \;\;249 \\
\hline
866) & 6068 \\
 & \;\;5196 \\
\hline
 & \;\;\;872
\end{array}
$$

from which it is obvious that the quantity proposed is not an exact square, but exceeds the square of 436 by the remainder 872: and it may be here observed that this remainder 872 is greater than the quantity 866 used as the last divisor, though the root is properly extracted, it being readily seen that the square of the next superior number 437 would exceed the quantity proposed by 1.

To find however the limit of the remainder after extracting the square root of any numerical magnitude, we observe, as before, that if a number be increased by 1, its square will be increased by twice the said number + 1; whence it follows that the remainder after any step of the operation

must be less than twice the corresponding number in the root + 1.

Thus, in the last example the remainder 872 is evidently less than $436 \times 2 + 1$ or 873, and from this we correctly conclude that 436 is the number whose square is next less than 190968, the number proposed.

31. Since the product of two quantities contains as many decimal places as are comprised in the multiplier and multiplicand together, it follows that in every quantity considered as a square, the number of decimal places must be even: and consequently the number of decimal places being rendered even if not so already, and the points placed as before directed, the numbers of points over the whole numbers and decimals proposed will indicate the corresponding numbers of figures in the whole numbers and decimals which compose the root.

Ex. To find the square root of 16489.1281.

First, pointing the quantity according to the directions previously given, we see that the root consists of three whole numbers and two decimals: thus,

```
        .  .  . .  .
        16489.1281 (128.41 = the square root.
        1
        ---------
   22)  64
        44
        ---------
  248)  2089
        1984
        ----------
 2564)  10512
         10256
         ---------
25681)   25681
          25681
          --------
```

32. Since the cube of $a+b = a^3 + 3a^2b + 3ab^2 + b^3$, we may similarly deduce a method of extracting the cube roots of numerical magnitudes by considering in what manner $a+b$ may be obtained from this last quantity. Thus,

Ex. 1. To extract the cube root of 13824, we have the following operation:

$$
\begin{array}{lrl}
 & & 13824\ (20+4=24 \text{ the cube root,} \\
a^3 & = & 8000 = \text{first subtrahend} \\
\text{also, } 3a^2 & = & 1200)\ 5824 = \text{first remainder} \\
\therefore\ 3a^2b & = & 4800 \\
3ab^2 & = & 960 \\
b^3 & = & 64 \\
 & & 5824 = \text{second subtrahend}
\end{array}
$$

so that 0 remains: and in this we enquired what was the greatest cube number next less than 13, omitting the last three figures of the quantity proposed: thus $a$ is obtained $=20$, and its cube being subtracted from the quantity proposed there remains 5824: the divisor being then made $= 3a^2$ which in this case is 1200, we next find that it is contained in the said remainder four times, so that $b=4$: we have then to obtain the next subtrahend which $= 3a^2b + 3ab^2 + b^3 = 5824$ in this instance, and after subtraction there is no remainder, the required cube root being thus extracted which $= 20+4$ or 24.

Ex. 2. To extract the cube root of 1860867, we have

$$
\begin{array}{lrl}
 & & 1860867\ (100+20+3=123; \\
a^3 & = & 1000000 = \text{first subtrahend} \\
\text{also, } 3a^2 & = & 30000)\ 860867 = \text{first remainder}
\end{array}
$$

$$\begin{array}{rlr} \therefore\ 3a^2b & = & 600000 \\ 3ab^2 & = & 120000 \\ b^3 & = & 8000 \\ \hline & & 728000 = \text{second subtrahend} \\ \hline \text{again, } 3a'^2 & = 43200) & 132867 = \text{second remainder} \\ \hline \therefore\ 3a'^2b' & = & 129600 \\ 3a'b'^2 & = & 3240 \\ b'^3 & = & 27 \\ \hline & & 132867 = \text{third subtrahend,} \\ \hline \end{array}$$

and then there is no remainder.

Here, omitting six figures towards the right of the number proposed, we know immediately that $a = 100$, and there remains after subtraction 860867: now the first divisor being $3a^2=30000$, we obtain $b=20$, and the subtrahend $3a^2b+3ab^2+b^3$ is then found $= 728000$, which leaves 132867 for the second remainder: again considering $a'$ equivalent to $a+b$ or 120, we have the second divisor $= 3a'^2 = 43200$, which is obviously contained three times in the said remainder: whence if the third figure 3 of the root be called $b'$, we have the third subtrahend $= 3a'^2b' + 3a'b'^2 + b'^3 = 132867$, after which there is no remainder left, the root being 123.

The same mode of proceeding may manifestly be extended to any number of figures whatever.

33. The operation of extracting the cube root may however be somewhat abbreviated by the following considerations.

$$\begin{array}{rl} \text{Since the cube of } 10 & = 1000, \\ \text{the cube of } 100 & = 1000000, \\ \text{the cube of } 1000 & = 1000000000, \\ \&c. \ldots\ldots\ldots & = \&c. \end{array}$$

it is a necessary consequence that the cube roots of numbers of less than four, seven, ten, &c. figures will consist of one, two, three, &c. figures respectively; and therefore if a point be placed over every third figure in the number proposed, beginning at the unit's place, the number of such points will indicate the number of figures in the root, and then the cyphers may be suppressed; thus, the operation of the last example may stand as below:

$$
\begin{array}{rrcrl}
 & & & \dot{1}86\dot{0}86\dot{7} & (123 = \text{the cube root.} \\
 & a^3 & = & 1 & = \text{first subtrahend} \\
\hline
\text{also,} & 3a^2 & = & 3)\ 860 & = \text{first remainder} \\
\therefore & 3a^2b & = & 6.. & \\
 & 3ab^2 & = & 12. & \\
 & b^3 & = & 8 & \\
\hline
 & & & 728 & = \text{second subtrahend} \\
\hline
\text{again,} & 3a'^2 & = & 432)\ 132867 & = \text{second remainder} \\
\therefore & 3a'^2b' & = & 1296.. & \\
 & 3a'b'^2 & = & 324. & \\
 & b'^3 & = & 27 & \\
\hline
 & & & 132867 & = \text{third subtrahend.} \\
\hline
\end{array}
$$

If the number of decimals in any quantity proposed be made a multiple of 3, the same plan in pointing and extracting the root may be employed; and at any step of the operation, as appears from example (8) of (22), the remainder must be less than three times the square of the corresponding quantity in the root together with three times the quantity itself + 1, if the figures in the root be so far properly obtained.

A course analogous to this might be pursued in extracting the higher roots of numbers, but the operation would in general be so complicated, that recourse is had to other expedients hereafter to be explained.

34. Let $x$ and $y$ represent any two quantities whatever, whereof $x$ is the greater and $y$ the less, then will $x+y$ and $x-y$ indicate their sum and difference respectively, as appears from (4) and (5):

$$\text{now } (x+y)+(x-y)=2x, \text{ by (18)},$$
$$\text{and } (x+y)-(x-y)=2y, \text{ by (19)}:$$

that is, the sum of any two quantities increased by their difference is equal to twice the greater: and the sum of any two quantities diminished by their difference is equal to twice the less.

Hence the greater of two quantities is equal to half their sum increased by half their difference, and the less is equal to half their sum diminished by half their difference.

35. Using the same notation, we shall have

$$(x+y)\times(x-y)=x^2-y^2, \text{ by (20)}:$$

that is, the product of the sum and difference of any two quantities is equal to the difference of the squares of the same quantities.

Hence also, if the difference of the squares of any two quantities be divided by the sum and difference of the same quantities, the quotients will be the difference and sum of those quantities respectively.

36. On the same hypothesis, we have

$$(x^3+y^3)\div(x+y)=x^2-xy+y^2,$$
$$\text{and } (x^3-y^3)\div(x-y)=x^2+xy+y^2, \text{ by (21)}:$$

whence it appears that if the sum and difference of the cubes of any two quantities be divided respectively by the sum and difference of the quantities themselves, the quotients will be the sum of the squares of the same quantities diminished or increased respectively by their product.

Hence also, if the sum of the squares of two quantities, diminished or increased by their product, be multiplied by the sum or difference of the same quantities, the product will be the sum or difference of their cubes.

37. We have seen in example (12) of article (21), that $a^m - x^m$ is always divisible by $a - x$ whatever positive whole number $m$ may be; wherefore since $a^m + x^m = a^m - x^m + 2\,x^m$, it follows that $a^m + x^m$ is not so divisible, but there is left a remainder $2\,x^m$.

Again, if for $x$ we substitute $-x$, we shall have to put $x^m$ in the place of $x^m$ when $m$ is even, and $-x^m$ when $m$ is odd, so that $a^m - x^m$ is generally divisible by $a + x$ when $m$ is even, and $a^m + x^m$ is so divisible when $m$ is odd.

Whence also $a^m + x^m$ is not divisible by $a + x$ when $m$ is even, nor is $a^m - x^m$ divisible by $a + x$ when $m$ is odd.

38. Since $(x + y)^2 = x^2 + 2\,xy + y^2$,

and $(x - y)^2 = x^2 - 2\,xy + y^2$, by (22);

we observe that the square of the sum or difference of any two quantities is equal to the sum of the squares of the quantities themselves increased or diminished by twice their product.

Also, because $(\pm 2\,xy)^2 = 4\,x^2y^2 = 4 \times x^2 \times y^2$, it is discovered that when any trinomial is a complete square, the square of the middle term is equal to four times the product of the extreme terms.

And because the square of every quantity, whether positive or negative, is positive, it is obvious that $x^2 - 2\,xy + y^2$ is

positive, and therefore $x^2 + y^2$ is greater than $2xy$: in other words, the sum of the squares of two unequal magnitudes is always greater than twice their product.

39. Since
$$\begin{aligned}(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\\ &= x^3 + y^3 + 3xy(x+y),\end{aligned}$$
and
$$\begin{aligned}(x-y)^3 &= x^3 - 3x^2y + 3xy^2 - y^3\\ &= x^3 - y^3 - 3xy(x-y), \text{ by (22)},\end{aligned}$$

we learn generally, that the cubes of the sum and difference of any two quantities are respectively equal to the sum and difference of the cubes of the same quantities, increased and diminished by three times their product multiplied by their sum and difference.

And $3x^2y \times 3xy^2$ being equal to $9x^3y^3$, it is thus demonstrated that in every quadrinomial which is a complete cube, the product of the mean terms is equal to nine times the product of the extreme terms.

40. If $a = x + y$, and both sides be multiplied by one of the latter quantities as $x$, we have

$$ax = x^2 + xy = xy + x^2:$$

that is, if any quantity be divided into two parts, the product of the whole and one of the parts is equal to the product of the two parts, together with the square of the other part.

41. Again, if $a = x + y$, we shall have $a^2 = x^2 + 2xy + y^2$,
$$\begin{aligned}\therefore\ a^2 + x^2 &= 2x^2 + 2xy + y^2\\ &= 2x(x+y) + y^2 = 2ax + y^2:\end{aligned}$$
which shews us, that if a quantity be divided into any two parts, the squares of the whole and of one of the parts, are together equal to twice the product of the whole and that part, together with the square of the other part.

42. If $2a$ be any proposed quantity, and $x$ be one part of it, then the remaining part is $2a-x$: and it is obvious that

$$(2a-x)x+(a-x)^2=2ax-x^2+a^2-2ax+x^2=a^2:$$

whence we infer that if a quantity be divided into two parts, the product of the two parts, together with the square of the excess of half the quantity above the less part is equal to the square of half the quantity itself.

43. On the same hypothesis as in the last article, we have

$$x^2+(2a-x)^2=x^2+4a^2-4ax+x^2$$
$$=4a^2-4ax+2x^2=2(2a^2-2ax+x^2)$$
$$=2(a^2+a^2-2ax+x^2)=2a^2+2(a-x)^2:$$

whence it results that if a quantity be divided into any two parts, the squares of the two parts are together double of the square of half the quantity, and of the square of the excess of its half above the less part.

44. If we have the equation $ax^2-bx+c=d$, then by annexing to both sides the quantity $bx-c$, the equation will become

$$ax^2-bx+c+bx-c=d+bx-c:$$

and observing that $-bx+bx$ and $+c-c$ are both $=0$, we have remaining

$$ax^2=d+bx-c,$$

which teaches us that any quantity may be *transposed* from one side of an equation to the other, merely by changing its algebraical sign from $+$ to $-$, or from $-$ to $+$.

In the same manner if $ax^2-bx+c$, be greater or less than $d$, it will follow that

$$ax^2 \text{ is greater or less than } d+bx-c:$$

and similar conclusions will hold good when both sides are equally affected by the operations of Multiplication, Division, Involution or Evolution.

# CHAP. III.

## *On the greatest Common Measures, and least Common Multiples of two or more Algebraical Quantities.*

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### I. COMMON MEASURES.

45. Def. A *Common Measure* of two or more quantities is a common divisor or quantity which divides them exactly, without leaving any remainder; and the greatest common measure is the greatest quantity by which they are so divisible.

Thus, of the quantities $2abd$ and $2dxy$, the factors $2$, $d$ and $2d$ are all common measures, the greatest being manifestly $2d$: and $2d$ is said to measure $2abd$ and $2dxy$, by the units in $ab$ and $xy$ respectively.

Similarly, of the quantities $abcd$, $adcy$ and $abdx$, $a$, $d$ and $ad$ are all common measures; and $ad$ is the greatest in the sense intended in the definition, without reference to the numerical values that might be assigned to $a$ and $d$.

46. Cor. 1. The greatest common measure of $ad$ and $bd$ is $d$, which is manifestly also the greatest common measure of $acd$ and $bd$: that is, the greatest common measure of two quantities is the greatest common measure of either of them, and the other multiplied or divided by any quantity which is not a divisor of the first, and which contains no factor common to them both.

47. Cor. 2. If $-d$ be a common divisor of any number of quantities, it is obvious that $d$ will also divide them without remainders; and hence it follows that the greatest common measure is always a positive quantity.

48. When the quantities proposed are monomials, the common measure is readily discovered by inspection, and the same method is applicable to many other cases of which some of the following are examples.

Ex. 1. The greatest common measure of $10ax$ and $15x^2$ is $5x$, the quotients being respectively $2a$ and $3x$.

Ex. 2. The greatest common measure of $8a^2xy$, $-12bxy^2$ and $20cx^2y$, is $4xy$, the respective quotients being $2a^2$, $-3by$ and $5cx$.

Ex. 3. The greatest common measure of $a^2-2ax+x^2$ and $ax-x^2$ is $a-x$, for these quantities are equivalent to $(a-x)^2$ and $x(a-x)$, whereof the greatest common divisor is $a-x$, the quotients being $a-x$ and $x$ respectively.

Ex. 4. The greatest common measure of $3(a+b)^3(c-x)^4$ and $5(a+b)^2(c-x)^5$ is manifestly $(a+b)^2(c-x)^4$, and the quotients will be $3(a+b)$ and $5(c-x)$.

Ex. 5. The greatest common measure of $a(x+y-z)^m$ and $b(x+y-z)^n$ will evidently be $(x+y-z)^n$ or $(x+y-z)^m$, according as $m$ is greater or less than $n$, and the corresponding quotients will be $a(x+y-z)^{m-n}$ and $b$, or $a$ and $b(x+y-z)^{n-m}$.

49. In all instances similar to those just given, there can never exist much difficulty in determining the greatest common measure, and whenever quantities can be reduced to the above-mentioned forms, their common measure will in general be manifest. Thus,

Ex. 1. If it be required to find the greatest common measure of $a^3+a^2b-ab^2-b^3$ and $a^3-a^2b-ab^2+b^3$, we have

$$a^3+a^2b-ab^2-b^3=(a^3+a^2b)-(ab^2+b^3)$$
$$=a^2(a+b)-b^2(a+b)=(a^2-b^2)(a+b),$$
$$\text{and } a^3-a^2b-ab^2+b^3=(a^3-a^2b)-(ab^2-b^3)$$
$$=a^2(a-b)-b^2(a-b)=(a^2-b^2)(a-b);$$

and since it is clear that $a+b$ and $a-b$ contain no common factor, the required greatest common measure is $a^2-b^2$.

Ex. 2. Of the two quantities $3bcz+5mxz+30mx+18bc$, and $4adz - 7vrz + 24ad - 42vr$, we observe that

$$\text{the former} = 3bc(z+6) + 5mx(z+6)$$
$$= (3bc+5mx)(z+6),$$

$$\text{and the latter} = 4ad(z+6) - 7vr(z+6)$$
$$= (4ad-7vr)(z+6);$$

and therefore, as in the last example, the greatest common measure is $z+6$.

50. In the generality of instances that occur, it is no easy matter to decompose the quantities into their factors as in the examples just given, and indeed in many cases this would be almost impossible. On this account it will be expedient to endeavour to devise some method which may be applicable to all quantities whatever.

51. *To investigate a rule for finding the greatest common measure of two quantities.*

Let $a$ and $b$ be the two quantities whereof $a$ is the greater, and let $b$ be contained $p$ times in $a$ with a remainder $c$: let $c$ be contained $q$ times in $b$ with a remainder $d$, and let $d$ be contained $r$ times in $c$ with no remainder, the operations being performed as under:

$$\begin{array}{llll}
b)\,a\,(p & & & \\
\;\;pb & & & \\
\hline
 & c)\,b\,(q & & \\
 & \;\;qc & & \\
\hline
 & & d)\,c\,(r & \\
 & & \;\;rd & \\
\hline
 & & \;\;\;0 &
\end{array}$$

then is $d$ the greatest common measure of $a$ and $b$.

For, since $c - rd = 0$, we have $c = rd$, by (44):

also $b - qc = d$, $\therefore\ b = d + qc = d + qrd = (1 + qr)\,d$:

and

$a - pb = c$, $\therefore\ a = c + pb = rd + p\,(1 + qr)\,d = (p + pqr + r)\,d$,

from which it appears that $d$ measures $a$ and $b$ by the units in $p + pqr + r$ and $1 + qr$ respectively, and is therefore a common measure.

It is, moreover, the greatest common measure; for, if not, let $D$ be the greatest common measure, and let it be contained $m$ and $n$ times respectively in $a$ and $b$, so that

$$a = mD \text{ and } b = nD;$$

$$\therefore\ c = a - pb = mD - npD = (m - np)\,D,$$

and $d = b - qc = nD - q\,(m - np)\,D = (n - mq + npq)\,D$,

wherefore $D$ measures $d$, or a greater quantity measures a less, which is absurd; $\therefore$ no quantity but $d$ is the greatest common measure; and the quotients arising from the division of $a$ and $b$ by $d$ are $p + pqr + r$ and $1 + qr$ respectively.

52. Cor. 1. Every common measure of $a$ and $b$ is a measure of the greatest common measure $d$.

For, let $\delta$ be any common measure of $a$ and $b$, so that

$$a = \mu\delta \text{ and } b = \nu\delta;$$

$$\therefore\ c = a - pb = \mu\delta - \nu p\delta = (\mu - \nu p)\,\delta,$$

and $d = b - qc = \nu\delta - q\,(\mu - \nu p)\,\delta = (\nu - \mu q + \nu pq)\,\delta$,

and therefore $\delta$ measures $d$ by the units in $\nu - \mu q + \nu pq$.

53. Cor. 2. From the nature of division it is obvious that each of the remainders $c$, $d$, &c. is less than that which immediately precedes it, and consequently that in every case the

division may be continued till the remainder becomes less than any quantity that can be assigned.

54. Cor. 3. In the demonstration of the proposition, it appears that if a quantity measure two or more others, it will also measure any expression formed out of them by the operations of addition, subtraction and multiplication.

55. From the operation exhibited in (51) we have the following general Rule for finding the greatest common measure of two quantities.

Arrange both the quantities according to the dimensions of some letter contained in them; divide the greater of them by the less, and the preceding divisor by the last remainder, and continue the operation till there is no remainder; then will the last divisor be the greatest common measure.

Ex. 1. To find the greatest common measure of $x^2+2x+1$ and $x^3+2x^2+2x+1$, we have by the rule the following operation:

$$
\begin{array}{l}
x^2+2x+1)\ x^3+2x^2+2x+1\ (x, \\
\qquad\qquad\quad \underline{x^3+2x^2+\ \ x\qquad} \\
\qquad\qquad\qquad\qquad x+1)\ x^2+2x+1\ (x+1, \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2+x\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x+1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x+1\ \ }
\end{array}
$$

whence $x+1$, being the last divisor, is the greatest common measure, and the quotients are $x+1$ and $x^2+x+1$.

Ex. 2. Find the greatest common measure of

$$x^4+x^2y^2+y^4 \text{ and } x^4+2x^3y+3x^2y^2+2xy^3+y^4.$$

Here

$$
\begin{array}{l}
x^4+x^2y^2+y^4)\ x^4+2x^3y+3x^2y^2+2xy^3+y^4\ (1, \\
\qquad\qquad\qquad\ \ \underline{x^4 \qquad\ \ +\ x^2y^2 \qquad\quad +y^4} \\
\qquad\qquad\qquad\qquad\ 2x^3y+2x^2y^2+2xy^3;
\end{array}
$$

now this remainder is equivalent to $2xy\,(x^2+xy+y^2)$, whereof neither $2xy$ nor any factor of it is a common measure, and therefore by (46), $2xy$ may be rejected, and the remaining factor $x^2+xy+y^2$ alone retained:

again,

$$
\begin{array}{l}
x^2+xy+y^2)\ x^4+x^2y^2+y^4\ (x^2-xy+y^2, \\
\qquad\qquad\quad\ \underline{x^4+x^3y+x^2y^2} \\
\qquad\qquad\qquad -x^3y+y^4 \\
\qquad\qquad\qquad \underline{-x^3y-x^2y^2-xy^3} \\
\qquad\qquad\qquad\qquad\quad x^2y^2+xy^3+y^4 \\
\qquad\qquad\qquad\qquad\quad \underline{x^2y^2+xy^3+y^4}
\end{array}
$$

wherefore $x^2+xy+y^2$ is the greatest common measure.

Ex. 3. Required the greatest common measure of
$8x^3-4x^2-2x+1$ and $12x^3+4x^2-3x-1$.

In this case, the first term of neither of the quantities is contained an exact number of times in the other, and since fractional quotients are excluded in the general proof, if we multiply the latter by 2, the common measure will not be altered, as appears from (46), and we have

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad\quad 12x^3+4x^2-3x-1 \\
\qquad\qquad\qquad\qquad\quad \underline{2 \qquad\qquad\qquad\qquad\quad} \\
8x^3-4x^2-2x+1)\ 24x^3+8x^2-6x-2\ (3, \\
\qquad\qquad\qquad\qquad\quad \underline{24x^3-12x^2-6x+3} \\
\qquad\qquad\qquad\qquad\qquad\qquad 20x^2-5;
\end{array}
$$

and this remainder is equivalent to $5\,(4\,x^2-1)$ whereof the former factor may manifestly be rejected by (46): therefore,

$$4\,x^2-1)\;8\,x^3-4\,x^2-2\,x+1\;(2\,x-1,$$
$$\begin{array}{r} 8\,x^3-2\,x \\ \hline -4\,x^2+1 \\ -4\,x^2+1 \\ \hline \end{array}$$

so that $4\,x^2-1$ is the greatest common measure.

Ex. 4. To find the greatest common measure of $x^3-5\,x^2-2\,x+10$ and $x^3+5\,x^2-2\,x-10$, we have

$$x^3+5\,x^2-2\,x-10)\;x^3-5\,x^2-2\,x+10\;(1,$$
$$\begin{array}{r} x^3+5\,x^2-2\,x-10 \\ \hline -10\,x^2+20=-10\,(x^2-2): \end{array}$$

again, $x^2-2)\;x^3+5\,x^2-2\,x-10\;(x+5,$

$$\begin{array}{r} x^3-2\,x \\ \hline 5\,x^2-10 \\ 5\,x^2-10 \\ \hline \end{array}$$

whence $x^2-2$ is the required common measure.

Ex. 5. To find the greatest common measure of

$2\,x^5-4\,x^4+8\,x^3-12\,x^2+6\,x$ and $3\,x^5-3\,x^4-6\,x^3+9\,x^2-3\,x$,

we observe in the first place, that

$2\,x^5-4\,x^4+8\,x^3-12\,x^2+6\,x=2\,x\,(x^4-2\,x^3+4\,x^2-6\,x+3)$

and $3\,x^5-3\,x^4-6\,x^3+9\,x^2-3\,x=3\,x\,(x^4-x^3-2\,x^2+3\,x-1)$:

now, it is obvious that $x$ is a common measure of the two quantities, but that 2 and 3 are not; ∴ it is next required

to find the greatest common measure of the remaining factors: thus,

$$x^4 - 2x^3 + 4x^2 - 6x + 3)\ x^4 - x^3 - 2x^2 + 3x - 1\ (1,$$
$$x^4 - 2x^3 + 4x^2 - 6x + 3$$
$$\overline{\quad x^3 - 6x^2 + 9x - 4:}$$

$$x^3 - 6x^2 + 9x - 4)\ x^4 - 2x^3 + 4x^2 - 6x + 3\ (x + 4,$$
$$x^4 - 6x^3 + 9x^2 - 4x$$
$$\overline{4x^3 - 5x^2 - 2x + 3}$$
$$4x^3 - 24x^2 + 36x - 16$$
$$\overline{19x^2 - 38x + 19,}$$

and this $= 19(x^2 - 2x + 1)$, whereof 19 may by (46) be rejected:

$$\therefore\ x^2 - 2x + 1)\ x^3 - 6x^2 + 9x - 4\ (x - 4,$$
$$x^3 - 2x^2 + x$$
$$\overline{-4x^2 + 8x - 4}$$
$$-4x^2 + 8x - 4$$
$$\overline{\qquad\qquad}$$

whence $x^2 - 2x + 1$ being the greatest common measure of the latter factors, the greatest common measure of the proposed quantities will manifestly be $x(x^2 - 2x + 1)$ or $x^3 - 2x^2 + x$.

56. *To find the greatest common Measure of three or more quantities.*

Let $a$, $b$, $c$ be any three algebraical quantities, and let $d$ be the greatest common measure of $a$ and $b$, then will the greatest common measure of $d$ and $c$ be the greatest common measure of $a$, $b$ and $c$.

For, since $d$ is the greatest common measure of $a$ and $b$, every measure of $d$ is a common measure of $a$ and $b$; therefore every common measure of $d$ and $c$ is a common measure of

$a$, $b$ and $c$, and the greatest common measure of $d$ and $c$ is the greatest common measure of $a$, $b$ and $c$.

In the same manner, whatever be the number of quantities, their greatest common measure may be determined by a continuation of the above mentioned process.

Ex. Find the greatest common measure of the three quantities $a^3 + a^2b - ab^2 - b^3$, $a^3 - 2a^2b - ab^2 + 2b^3$ and $a^3 - 3ab^2 + 2b^3$.

In the first place, to find the greatest common measure of the first two quantities, we have

$$\begin{array}{l} a^3 + a^2b - ab^2 - b^3)\ a^3 - 2a^2b - ab^2 + 2b^3\ (1, \\ \qquad\qquad\qquad\qquad \underline{a^3 + \ a^2b - ab^2 - \ b^3} \\ \qquad\qquad\qquad\qquad\quad -3a^2b + 3b^3 = -3b(a^2 - b^2), \end{array}$$

of which the factor $-3b$ being rejected, we have

$$\begin{array}{l} a^2 - b^2)\ a^3 + a^2b - ab^2 - b^3\ (a - b, \\ \qquad\quad \underline{a^3 - ab^2} \\ \qquad\qquad\quad a^2b - b^3 \\ \qquad\qquad\quad \underline{a^2b - b^3} \end{array}$$

so that $a^2 - b^2$ is the greatest common measure of the first two quantities, and it remains to find the same of this and the third:

$$\begin{array}{l} a^2 - b^2)\ a^3 - 3ab^2 + 2b^3\ (a, \\ \qquad\quad \underline{a^3 - \ ab^2} \\ \qquad\qquad -2ab^2 + 2b^3 = -2b^2(a - b), \end{array}$$

which, the former factor being rejected, gives

$$\begin{array}{l} a - b)\ a^2 - b^2\ (a + b, \\ \qquad\quad \underline{a^2 - ab} \\ \qquad\qquad\ ab - b^2 \\ \qquad\qquad\ \underline{ab - b^2} \end{array}$$

wherefore $a-b$ is the greatest common measure of the three quantities proposed, and it is contained in them $a^2+2ab+b^2$, $a^2-ab-2b^2$ and $a^2+ab-2b^2$ times respectively.

## II. COMMON MULTIPLES.

57. Def. A *Common Multiple* of two or more quantities is a common dividend or quantity which contains each of them an exact number of times; and the least common multiple is the least quantity which they can each divide without a remainder.

Thus, $2abc$ is a common multiple of $ab$ and $bc$, and $abc$ is their least common multiple: so also $3abx$ is the least common multiple of the quantities $3a$, $3bx$ and $abx$.

58. Cor. Hence the least common multiple of any number of quantities, having no common measure except unity, is their product.

59. The least common multiples of monomials and other quantities involving common measures that are apparent may generally be found with ease by inspection, as in the following instances.

Ex. 1. The least common multiple of $a^2bc$ and $2ab^2d$, is $2a^2b^2cd$.

Ex. 2. The least common multiple of $axy$, $a^2y$ and $ax+by$, is $a^3x^2y+a^2bxy^2$.

Ex. 3. The least common multiple of $a^2(x+y)$, $ab(x-y)$ and $x^2-y^2$, is $a^2b(x^2-y^2)$.

Ex. 4. The least common multiple of $(a-b)^2(c+x)^2$ and $(a-b)^3(c+x)$, is $(a-b)^3(c+x)^2$.

Ex. 5. The least common multiple of $ab+cd$, $ab-cd$ and $a^2b^2+c^2d^2$, is $a^4b^4-c^4d^4$.

60. *To investigate a rule for finding the least common Multiple of two quantities.*

Let $a$ and $b$ be the two quantities, $d$ their greatest common measure such that $a = pd$ and $b = qd$, and $m$ their least common multiple; then, since $p$ and $q$ have no common measure except unity, their least common multiple is $pq$, by (58): wherefore $pqd$ will manifestly be the least common multiple of $pd$ and $qd$, or of $a$ and $b$: that is, the least common multiple $m$ of $a$ and $b$ is $pqd = (pd \times qd) \div d = (a \times b) \div d$, and the numbers of times it contains them are $(a \div d)$ and $(b \div d)$ respectively.

61. Cor. Every common multiple of $a$ and $b$ is a multiple of the least common multiple $m$.

For, let $\mu$ be any common multiple of $a$ and $b$, and if possible, let $m$ be contained $r$ times in $\mu$ with a remainder $s$, so that

$$\mu = rm + s; \ \therefore \text{ by (44), } s = \mu - rm:$$

wherefore since $a$ and $b$ measure $\mu$ and $m$, they will also by (54) measure $s$ which is less than $m$; that is, $m$ is not the least common multiple of $a$ and $b$, contrary to the supposition: hence every other common multiple is a multiple of the least common multiple.

62. The result of the investigation contained in (60) furnishes the following Rule for finding the least common multiple of any two quantities.

Find the product of the two quantities, divide it by their greatest common measure, and the result will be the least common multiple.

Ex. 1. Required the least common multiple of $a^3 + a^2b$ and $a^2 - b^2$.

The greatest common measure of $a^3 + a^2b$ and $a^2 - b^2$ being $a + b$, we shall have their least common multiple

$$= (a^3 + a^2b) \times (a^2 - b^2) \div (a + b) = a^2 \times (a^2 - b^2) = a^4 - a^2b^2.$$

Ex. 2. Find the least common multiple of $x^3+x^2+x+1$ and $x^3-x^2+x-1$.

Here, we have in the first place,

$$x^3+x^2+x+1=x(x^2+1)+(x^2+1)=(x+1)(x^2+1),$$

and $x^3-x^2+x-1=x(x^2+1)-(x^2+1)=(x-1)(x^2+1)$;

therefore the least common multiple

$$=(x+1)(x-1)(x^2+1)=(x^2-1)(x^2+1)=x^4-1.$$

63. *To find the least common Multiple of three or more quantities.*

Let $a$, $b$, $c$ be three proposed quantities, and let $m$ be the least common multiple of $a$ and $b$, then the least common multiple of $m$ and $c$ is the least common multiple of $a$, $b$ and $c$.

For, since $m$ is the least common multiple of $a$ and $b$, every multiple of $m$ is a common multiple of $a$ and $b$, and every common multiple of $m$ and $c$ is a common multiple of $a$, $b$ and $c$; whence it follows that the least common multiple of $m$ and $c$ is the least common multiple of $a$, $b$ and $c$.

The same kind of reasoning is applicable, whatever be the number of quantities proposed.

Ex. Required the least common multiple of $a^2+ab$, $a^4+a^2b^2$ and $a^4-b^4$.

Since $a^2+ab=a(a+b)$ and $a^4+a^2b^2=a^2(a^2+b^2)$,

we have the least common multiple of the first two

$$=(a+b)\times a^2(a^2+b^2)=a^2(a+b)(a^2+b^2):$$

and since $a^4-b^4=(a^2-b^2)(a^2+b^2)=(a+b)(a-b)(a^2+b^2)$,

the least common multiple required will obviously be

$$=a^2\times(a^2-b^2)(a^2+b^2)=a^6-a^2b^4.$$

## CHAP. IV.

### *On Fractional Quantities.*

64. Def. An *Algebraical Fraction* is of the same nature as a fraction in common Arithmetic, and has an algebraical quantity for its numerator or denominator or both. Thus,

$$\frac{a}{2}, \quad \frac{a+x}{3}, \quad \frac{4}{b}, \quad \frac{5}{a^2-x^2}, \quad \frac{a}{b} \text{ and } \frac{x-3y}{5x+2y},$$

are all algebraical fractions.

#### I. REDUCTION.

65. *To represent an Integral Quantity as a Fraction.*

Let $a$ be any integral quantity, then it is obvious that by taking unity for a denominator, we shall have it equivalent to $\frac{a}{1}$: also, from the nature of fractions it appears that its value will not be altered by multiplying both the numerator and denominator by the same quantity, and thence we have

$$a = \frac{a}{1} = \frac{2a}{2} = \&c. = \frac{ad}{d} = \frac{-ad}{-d} = \&c.$$

66. Cor. 1. Since $\frac{ad}{d} = a = \frac{-ad}{-d}$, we infer that the value of a fraction is not altered by changing the signs of the numerator and denominator, which is in fact the same thing as multiplying each of them by $-1$.

67. Cor. 2. Conversely, a fraction is sometimes equivalent to, and may be represented by, an integral quantity. Thus,

$$\frac{ad}{d}=a,\ \frac{a^3-b^3}{a^2+ab+b^2}=a-b \text{ and } \frac{x^3+x^2y-xy^2-y^3}{(x+y)^2}=x-y.$$

68. *To represent a Mixed Quantity as a Fraction.*

Let $a+\frac{bc}{d}$ or $a-\frac{bc}{d}$ be any mixed quantity; then, since by (65), $a$ is equivalent to $\frac{ad}{d}$, we shall have $a\pm\frac{bc}{d}$ equivalent to $\frac{ad}{d}\pm\frac{bc}{d}$ or $\frac{ad\pm bc}{d}$.

Ex. 1. To reduce $a+\frac{(a-x)^2}{4x}$ to a fractional form, we have in the first place $a=\frac{4ax}{4x}$;

$$\therefore\ a+\frac{(a-x)^2}{4x}=\frac{4ax+(a-x)^2}{4x}=\frac{4ax+a^2-2ax+x^2}{4x}$$

$$=\frac{a^2+2ax+x^2}{4x}=\frac{(a+x)^2}{4x}.$$

Ex. 2. In the mixed quantity $a+b-\frac{2ab+b^2}{a+b}$, we shall have

$$a+b=\frac{(a+b)(a+b)}{a+b}=\frac{a^2+2ab+b^2}{a+b};$$

$$\therefore \text{ the equivalent fraction }=\frac{a^2+2ab+b^2-2ab-b^2}{a+b}=\frac{a^2}{a+b}.$$

Ex. 3. Again, $a+\frac{a^2+b^2-x^2}{2b}=\frac{2ab+a^2+b^2-x^2}{2b}$.

$$= \frac{(a+b)^2 - x^2}{2b} = \frac{(a+b+x)(a+b-x)}{2b}, \text{ and } a - \frac{a^2+b^2-x^2}{2b}$$

$$= \frac{2ab - a^2 - b^2 + x^2}{2b} = \frac{x^2 - (a-b)^2}{2b} = \frac{(x+a-b)(x+b-a)}{2b}.$$

69. Cor. Conversely, a fraction may sometimes be reduced to a mixed quantity, for $\frac{ad \pm bc}{d}$ is equivalent to $a \pm \frac{bc}{d}$.

Ex. 1. The fraction $\frac{a^2 + 4ab + 4b^2}{a}$ is equivalent to the mixed quantity $a + 4b + \frac{4b^2}{a}$, by actual division of the first two terms of the numerator by the denominator.

Ex. 2. The fraction $\frac{a^3 - b^3 + x^3}{a + x}$ by actual division, becomes equal to the mixed quantity $a^2 - ax + x^2 - \frac{b^3}{a+x}$.

Ex. 3. Similarly, the fraction

$$\frac{(a+b)^2 - (c-d)^2}{2(ab+cd)} = \frac{a^2 + 2ab + b^2 - c^2 + 2cd - d^2}{2(ab+cd)}$$

$$= \frac{2(ab+cd) + a^2 + b^2 - c^2 - d^2}{2(ab+cd)} = 1 + \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)},$$

a mixed quantity.

70. *To reduce a Fraction to its lowest Terms.*

Since, by the nature of fractions, $\frac{ad}{bd} = \frac{a}{b}$, it is obvious that a fraction may be reduced to its lowest terms by dividing both the numerator and denominator by their greatest common measure.

If $p$, $q$, $r$ be the quotients obtained in finding the greatest common measure of $a$ and $b$, then by (51), $\dfrac{p+pqr+r}{1+qr}$ will be the value of $\dfrac{a}{b}$ expressed in its lowest terms.

Ex. 1. The fraction $\dfrac{a^2xy^2}{4a^2x^2y}$ is reduced to its least terms $\dfrac{y}{4x}$, by dividing both the numerator and denominator by their greatest common measure $a^2xy$.

Ex. 2. The fraction

$$\frac{x^3-b^2x}{(x+b)^2}=\frac{x(x+b)(x-b)}{(x+b)^2}=\frac{x(x-b)}{x+b},$$

the greatest common measure being manifestly $x+b$.

Ex. 3. In the fraction $\dfrac{6a^2+\ 5ax-6x^2}{6a^2+13ax+6x^2}$, the greatest common measure is found by (55) to be $2a+3x$, and the numerator and denominator being divided by it, the value of the fraction, expressed in its lowest terms, is $\dfrac{3a-2x}{3a+2x}$.

Ex. 4. The fraction

$$\frac{30x^5+4x^4-20x^3+12x^2+6x}{36x^6+8x^4+16x^3-\ 8x^2-4x}$$

is equivalent to

$$\frac{2x(15x^4+2x^3-10x^2+6x+3)}{4x(\ 9x^5+2x^3+\ 4x^2-2x-1)}=\frac{15x^4+2x^3-10x^2+6x+3}{2(9x^5+2x^3+4x^2-2x-1)};$$

and the greatest common measure of the last numerator and the latter factor of the denominator being found by (55) to be $3x+1$, the fraction in its lowest terms is

$$\frac{5x^3-\ x^2-3x+3}{6x^4-2x^3+2x^2+2x-2}.$$

71. *To reduce Fractions to others having a common Denominator.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, then, by equal multiplication of their numerators and denominators respectively,

$$\text{we have } \frac{a}{b} = \frac{ad}{bd} \text{ and } \frac{c}{d} = \frac{bc}{bd};$$

therefore the proposed fractions are equivalent to

$\frac{ad}{bd}$ and $\frac{bc}{bd}$ which have the common denominator $bd$:

hence, the same kind of process being applicable whatever be the number of fractions proposed, we have merely to multiply every numerator by all the denominators except that which belongs to it, for a new numerator, and all the denominators together for a common denominator.

Ex. 1. To reduce $\frac{a}{2b}$, $\frac{3b}{a}$ and $\frac{5xy}{c}$ to other fractions with a common denominator, we have

$$\frac{a}{2b} = \frac{a \times a \times c}{2b \times a \times c} = \frac{a^2c}{2abc},$$

$$\frac{3b}{a} = \frac{3b \times 2b \times c}{2b \times a \times c} = \frac{6b^2c}{2abc},$$

$$\text{and } \frac{5xy}{c} = \frac{5xy \times 2b \times a}{2b \times a \times c} = \frac{10abxy}{2abc},$$

the resulting fractions having the common denominator $2abc$.

Ex. 2. To transform $\frac{a+x}{a-x}$, $\frac{a-x}{a+x}$ and $\frac{a^2-x^2}{a^2+x^2}$ so as to have a common denominator, we obtain

$$(a+x) \times (a+x) \times (a^2+x^2) = a^4 + 2a^3x + 2a^2x^2 + 2ax^3 + x^4,$$

$$(a-x) \times (a-x) \times (a^2+x^2) = a^4 - 2a^3x + 2a^2x^2 - 2ax^3 + x^4,$$

$$(a^2-x^2) \times (a-x) \times (a+x) = a^4 - 2a^2x^2 + x^4,$$

for the new numerators:

also, $(a-x)\times(a+x)\times(a^2+x^2)=a^4-x^4$, is the common denominator;

and hence the new equivalent fractions are

$$\frac{a^4+2a^3x+2a^2x^2+2ax^3+x^4}{a^4-x^4},\ \frac{a^4-2a^3x+2a^2x^2-2ax^3+x^4}{a^4-x^4}$$

$$\text{and}\ \frac{a^4-2a^2x^2+x^4}{a^4-x^4}.$$

Ex. 3. Reduce the mixed quantities

$$x+\frac{2ax+x^2}{2a-x},\ 2a-\frac{2ax}{2a+x}\ \text{and}\ 4x^2-a^2+\frac{a^4}{a^2+4x^2},$$

to fractions having a common denominator.

First, we have by article (68),

$$x+\frac{2ax+x^2}{2a-x}=\frac{2ax-x^2+2ax+x^2}{2a-x}=\frac{4ax}{2a-x},$$

$$2a-\frac{2ax}{2a+x}=\frac{4a^2+2ax-2ax}{2a+x}=\frac{4a^2}{2a+x},$$

$$4x^2-a^2+\frac{a^4}{a^2+4x^2}=\frac{16x^4-a^4+a^4}{a^2+4x^2}=\frac{16x^4}{a^2+4x^2}:$$

then

$4ax\times(2a+x)\times(a^2+4x^2)=8a^4x+4a^3x^2+32a^2x^3+16ax^4$,

$4a^2\times(2a-x)\times(a^2+4x^2)=8a^5-4a^4x+32a^3x^2-16a^2x^3$,

$16x^4\times(2a-x)\times(2a+x)=64a^2x^4-16x^6$,

which are the new numerators:

and $(2a-x)\times(2a+x)\times(a^2+4x^2)=4a^4+15a^2x^2-4x^4$, which is the common denominator;

wherefore the new equivalent fractions are

$$\frac{8a^4x+4a^3x^2+32a^2x^3+16ax^4}{4a^4+15a^2x^2-4x^4},\ \frac{8a^5-4a^4x+32a^3x^2-16a^2x^3}{4a^4+15a^2x^2-4x^4}$$

$$\text{and}\ \frac{64a^2x^4-16x^6}{4a^4+15a^2x^2-4x^4}.$$

72. Cor. If the denominators of the proposed fractions have a common measure, the fractions may be transformed into others having a less common denominator than what would have been determined by the preceding process.

Thus, if the fractions be $\frac{a}{bd}$ and $\frac{c}{de}$, we have immediately

$$\frac{a}{bd}=\frac{ae}{bde}\ \text{and}\ \frac{c}{de}=\frac{bc}{bde}:$$

from which it appears that the least common denominator is the least common multiple of the proposed denominators, and the numerators are obtained by multiplying the original numerators by the quotients arising from the division of the least common multiple by the corresponding denominators.

Ex. Reduce $\frac{x^2}{a^2+ax}$, $\frac{a^2}{ax-x^2}$ and $\frac{ax}{a^2-x^2}$ to equivalent fractions having the least common denominator.

By (63) we find the least common multiple of $a^2+ax$, $ax-x^2$ and $a^2-x^2$, to be $ax(a^2-x^2)$ or $a^3x-ax^3$, which is also the least common denominator: whence we have likewise

$$\frac{x^2\times ax(a^2-x^2)}{a(a+x)}=x^3(a-x)=ax^3-x^4,$$

$$\frac{a^2\times ax(a^2-x^2)}{x(a-x)}=a^3(a+x)=a^4+a^3x,$$

$$\frac{ax\times ax(a^2-x^2)}{a^2-x^2}=ax\times ax=a^2x^2,$$

which are the new numerators: and therefore the new equivalent fractions required are $\frac{ax^3-x^4}{a^3x-ax^3}$, $\frac{a^4+a^3x}{a^3x-ax^3}$ and $\frac{a^2x^2}{a^3x-ax^3}$.

## II. ADDITION.

73. *To find the Sum of two Fractions.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the proposed fractions, and assume $\frac{a}{b}=x$ and $\frac{c}{d}=y$; whence multiplying both sides by $b$ and $d$ respectively, we have $a=bx$ and $c=dy$:

$$\therefore\ ad=bdx \text{ and } bc=bdy,$$

consequently, $bdx+bdy$, or $bd(x+y)=ad+bc$,

$$\text{and } \therefore\ x+y, \text{ or } \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}:$$

from which it appears that the fractions are reduced to others having a common denominator, and the required sum is equal to the sum of the new numerators divided by the common denominator.

It is also evident that the process must be the same whatever be the number of fractions proposed.

Ex. 1. Find the sum of $\frac{a}{a+x}$ and $\frac{x}{a-x}$.

By reducing these fractions to others having a common denominator by (71), we have

$$\frac{a}{a+x}=\frac{a\times(a-x)}{(a+x)\times(a-x)}=\frac{a^2-ax}{a^2-x^2},$$

$$\text{and } \frac{x}{a-x}=\frac{x\times(a+x)}{(a-x)\times(a+x)}=\frac{ax+x^2}{a^2-x^2};$$

$$\therefore \text{ the required sum }=\frac{a^2-ax+ax+x^2}{a^2-x^2}=\frac{a^2+x^2}{a^2-x^2}.$$

Ex. 2. Find the sum of $x - \dfrac{2xy}{x+y}$ and $x + \dfrac{2xy}{x-y}$.

$$\text{First, } x - \frac{2xy}{x+y} = \frac{x^2 + xy - 2xy}{x+y} = \frac{x^2 - xy}{x+y} = x\left(\frac{x-y}{x+y}\right),$$

$$\text{and } x + \frac{2xy}{x+y} = \frac{x^2 - xy + 2xy}{x-y} = \frac{x^2 + xy}{x-y} = x\left(\frac{x+y}{x-y}\right);$$

∴ the required sum will obviously

$$= x\left(\frac{x-y}{x+y}\right) + x\left(\frac{x+y}{x-y}\right) = x\left\{\frac{x-y}{x+y} + \frac{x+y}{x-y}\right\}$$

$$= x\left\{\frac{x^2 - 2xy + y^2 + x^2 + 2xy + y^2}{x^2 - y^2}\right\} = \frac{2x(x^2+y^2)}{x^2-y^2}.$$

Ex. 3. Find the sum of the fractions $\dfrac{a}{x}$, $\dfrac{ax}{x^2-a^2}$ and $\dfrac{a^2}{x^2+a^2}$.

$$\left.\begin{array}{l} \text{First, } a \times (x^2 - a^2) \times (x^2 + a^2) = ax^4 - a^5, \\ ax \times \quad x \quad \times (x^2 + a^2) = ax^4 + a^3x^2, \\ a^2 \times \quad x \quad \times (x^2 - a^2) = a^2x^3 - a^4x, \end{array}\right\}\text{new numerators;}$$

and $x \times (x^2 - a^2) \times (x^2 + a^2) = x^5 - a^4x$, the common denominator; whence the fractions proposed are equivalent to

$$\frac{ax^4 - a^5}{x^5 - a^4x}, \quad \frac{ax^4 + a^3x^2}{x^5 - a^4x} \text{ and } \frac{a^2x^3 - a^4x}{x^5 - a^4x};$$

and their sum will therefore be

$$= \frac{ax^4 - a^5 + ax^4 + a^3x^2 + a^2x^3 - a^4x}{x^5 - a^4x}$$

$$= \frac{2ax^4 + a^2x^3 + a^3x^2 - a^4x - a^5}{x^5 - a^4x}.$$

Ex. 4. Let the proposed fractions be

$$\frac{1}{4a^3(a+x)},\quad \frac{1}{4a^3(a-x)}\ \text{and}\ \frac{1}{2a^2(a^2+x^2)}:$$

then it is obvious that their sum will be equivalent to

$$\frac{1}{4a^3}\left\{\frac{1}{a+x}+\frac{1}{a-x}\right\}+\frac{1}{2a^2(a^2+x^2)}$$

$$=\frac{1}{4a^3}\left\{\frac{2a}{a^2-x^2}\right\}+\frac{1}{2a^2(a^2+x^2)}$$

$$=\frac{1}{2a^2}\left\{\frac{1}{a^2-x^2}+\frac{1}{a^2+x^2}\right\}=\frac{1}{a^4-x^4}.$$

## III. SUBTRACTION.

74. *To find the Difference of two Fractions.*

Let the proposed fractions $\frac{a}{b}$ and $\frac{c}{d}$ be assumed equal to $x$ and $y$ respectively; then, as before, we shall have

$$a=bx,\ c=dy,\ \text{and}\ \therefore\ ad=bdx,\ bc=bdy:$$

whence $bdx-bdy$, or $bd(x-y)=ad-bc$,

$$\text{and}\ \therefore\ x-y,\ \text{or}\ \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}:$$

that is, the difference of two fractions is found by first reducing them to others having a common denominator, and then taking the difference of the new numerators and placing under it the common denominator.

Ex. 1. Find the difference of $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$.

$$\text{Here}\ \frac{x+y}{x-y}=\frac{(x+y)\times(x+y)}{(x-y)\times(x+y)}=\frac{x^2+2xy+y^2}{x^2-y^2},$$

$$\text{and}\ \frac{x-y}{x+y}=\frac{(x-y)\times(x-y)}{(x+y)\times(x-y)}=\frac{x^2-2xy+y^2}{x^2-y^2}:$$

whence the required difference $=\frac{4xy}{x^2-y^2}$.

Ex. 2. Let $\dfrac{a}{(a-b)(x+a)}$ and $\dfrac{b}{(a-b)(x+b)}$ be the proposed fractions; then their difference is manifestly

$$=\frac{1}{a-b}\left\{\frac{a}{x+a}-\frac{b}{x+b}\right\}$$

$$=\frac{1}{a-b}\left\{\frac{ax+ab-bx-ab}{(x+a)(x+b)}\right\}=\frac{x}{(x+a)(x+b)}.$$

Ex. 3. Required the difference of $\dfrac{a+x}{3a-2x}$ and $\dfrac{5a-2x}{2a-9x}$.

$$\text{First, } (a+x)\times(2a-9x)=2a^2-7ax-9x^2,$$

$$(5a-2x)\times(3a-2x)=15a^2-16ax+4x^2,$$

$$\text{and } (3a-2x)\times(2a-9x)=6a^2-31ax+18x^2,$$

so that the new equivalent fractions having a common denominator, are

$$\frac{2a^2-7ax-9x^2}{6a^2-31ax+18x^2} \text{ and } \frac{15a^2-16ax+4x^2}{6a^2-31ax+18x^2};$$

and taking the former from the latter, we have the remainder

$$=\frac{13a^2-9ax+13x^2}{6a^2-31ax+18x^2}.$$

Ex. 4. Find the difference of $x-\dfrac{x^2}{2a+x}$ and $a-\dfrac{a^2}{2x+a}$.

$$\text{First, by (68), } x-\frac{x^2}{2a+x}=\frac{2ax+x^2-x^2}{2a+x}=\frac{2ax}{2a+x},$$

$$\text{and } a-\frac{a^2}{2x+a}=\frac{2ax+a^2-a^2}{2x+a}=\frac{2ax}{2x+a}:$$

$\therefore$ we shall have the required difference $=\dfrac{2ax}{2a+x}-\dfrac{2ax}{2x+a}$

$$= 2ax\left\{\frac{1}{2a+x} - \frac{1}{2x+a}\right\} = 2ax\left\{\frac{2x+a-2a-x}{(2a+x)\,(2x+a)}\right\}$$

$$= \frac{2ax\,(x-a)}{(2a+x)\,(2x+a)} = \frac{2ax^2 - 2a^2x}{2a^2+5ax+2x^2}.$$

## IV. MULTIPLICATION.

75. *To find the Product of two Fractions.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the proposed fractions, and as before assume $\frac{a}{b} = x$ and $\frac{c}{d} = y$, so that $a = bx$ and $c = dy$:

$\therefore$ by multiplication, we have

$$a \times c = bx \times dy, \quad \text{or} \quad ac = bdxy,$$

$$\text{whence } xy, \text{ or } \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{a \times c}{b \times d}:$$

wherefore the product of two fractions is found by multiplying together their numerators and denominators respectively: and it is moreover obvious that the same rule holds of any number of fractions whatever.

Ex. 1. The product of $\frac{ax}{(a-x)^2}$ and $\frac{a^2-x^2}{ab}$ will be

$$\frac{ax \times (a^2-x^2)}{(a-x)^2 \times ab} = \frac{ax \times (a-x) \times (a+x)}{ab \times (a-x) \times (a-x)} = \frac{x\,(a+x)}{b\,(a-x)} = \frac{ax+x^2}{ab-bx},$$

by rejecting such factors as are common to the numerator and denominator.

Ex. 2. The product of $\frac{a^3-x^3}{a^3+x^3}$ and $\frac{(a+x)^2}{(a-x)^2}$ will be

$$\frac{(a-x)\times(a^2+ax+x^2)}{(a+x)\times(a^2-ax+x^2)}\times\frac{(a+x)\times(a+x)}{(a-x)\times(a-x)}$$

$$=\frac{(a^2+ax+x^2)\times(a+x)}{(a^2-ax+x^2)\times(a-x)}=\frac{a^3+2a^2x+2ax^2+x^3}{a^3-2a^2x+2ax^2-x^3},$$

by rejecting the common factors, and effecting the multiplications.

Ex. 3. Multiply $\dfrac{a-x}{a+x}-\dfrac{ax}{a^2-x^2}$ by $\dfrac{(a-x)(a+3x)}{a-3x}$.

First, $\dfrac{a-x}{a+x}-\dfrac{ax}{a^2-x^2}=\dfrac{(a-x)^2}{a^2-x^2}-\dfrac{ax}{a^2-x^2}=\dfrac{a^2-3ax+x^2}{a^2-x^2}$;

$\therefore$ the required product $=\dfrac{a^2-3ax+x^2}{a^2-x^2}\times\dfrac{(a-x)(a+3x)}{a-3x}$

$$=\frac{(a^2-3ax+x^2)\times(a+3x)}{(a+x)\times(a-3x)}=\frac{a^3-8ax^2+3x^3}{a^2-2ax-3x^2},$$ as before.

Ex. 4. Multiply $\dfrac{a}{x}+\dfrac{b}{y}$ by $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$.

In this case, we proceed as in integral quantities, and following the rules already given, shall have the operation as underneath:

$$\begin{array}{rll}
\text{the multiplicand} = & \dfrac{a}{x}+\dfrac{b}{y} & \\
\text{the multiplier} = & \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} & \\
\hline
& \dfrac{x}{a}+\dfrac{bx^2}{a^2y} & \\
& \quad\ \ \dfrac{ay^2}{b^2x}+\dfrac{y}{b} & \\
\hline
\therefore \text{ the product} = & \dfrac{x}{a}+\dfrac{bx^2}{a^2y}+\dfrac{ay^2}{b^2x}+\dfrac{y}{b}. &
\end{array}$$

The proposed fractions being equivalent to $\frac{ay+bx}{xy}$ and $\frac{b^2x^2+a^2y^2}{a^2b^2}$, their product will also $= \frac{(ay+bx)\,(b^2x^2+a^2y^2)}{a^2b^2xy}$, which, by performing the multiplication and requisite reductions, becomes the same as above.

76. Cor. 1. Since the product of $\frac{a}{bc}$ and $b$ is equivalent to $\frac{a}{bc} \times \frac{b}{1} = \frac{ab}{bc} = \frac{a}{c}$ by (70), we see that when a fraction is to be multiplied by an integral quantity, it is the same thing whether the numerator be multiplied by it, or the denominator be divided by it.

77. Cor. 2. A compound fraction, as $\frac{a}{b}$ of $\frac{c}{d}$, will, from the nature of fractions, manifestly be equal to the product of $\frac{a}{b}$ and $\frac{c}{d}$, or $\frac{ac}{bd}$: and thus a compound fraction may always be reduced to a simple one.

78. Cor. 3. Since, by definition (10), the quantities $a^{-m}$ and $a^{-n}$ are equivalent to $\frac{1}{a^m}$ and $\frac{1}{a^n}$ respectively, we shall have $$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}}, \text{ or } a^{-m-n};$$

again, $$a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ or } a^{m-n}:$$

and thus we perceive that Rule 2. of Article (20) is equally applicable, whether the indices be positive or negative, or partly positive and partly negative.

## V. DIVISION.

79. *To find the Quotient of two Fractions.*

Taking $\frac{a}{b}$ and $\frac{c}{d}$ for the proposed fractions, let us assume $\frac{a}{b} = x$ and $\frac{c}{d} = y$, as before; $\therefore\ a = bx$ and $c = dy$,

whence $ad = bdx$ and $bc = bdy$;

$\therefore$ by equal division, we have $\frac{bdx}{bdy} = \frac{ad}{bc}$:

that is, $\frac{x}{y}$, or $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}$:

wherefore the quotient of two fractions is found by taking the product of the dividend and the divisor inverted.

Ex. 1. Divide $\frac{2ax - x^2}{c^3 - x^3}$ by $\frac{2a - x}{(c-x)^2}$.

The quotient will here obviously be

$$= \frac{2ax - x^2}{c^3 - x^3} \div \frac{2a - x}{(c-x)^2} = \frac{(2a - x)\,x}{(c-x)\,(c^2 + cx + x^2)} \times \frac{(c-x)^2}{2a - x}$$

$= \frac{x\,(c-x)}{c^2 + cx + x^2} = \frac{cx - x^2}{c^2 + cx + x^2}$, by rejecting the factors common to both the numerator and denominator.

Ex. 2. If we divide $\frac{a^4 - b^4}{(a+2b)^2}$ by $\left(\frac{a^2 - b^2}{ab + 2b^2}\right)^2$, we shall have the quotient $= \frac{a^4 - b^4}{(a+2b)^2} \div \left(\frac{a^2 - b^2}{ab + 2b^2}\right)^2$

$$= \frac{(a^2 + b^2)\,(a^2 - b^2)}{(a+2b)^2} \times \frac{b^2\,(a+2b)^2}{(a^2 - b^2)^2} = \frac{b^2\,(a^2 + b^2)}{a^2 - b^2},$$

by proceeding as above.

Ex. 3. Divide $\frac{(a-x)^2}{x} + 2a$ by $\frac{(a+x)^2}{2ax} - 1$.

First, $\frac{(a-x)^2}{x} + 2a = \frac{a^2 - 2ax + x^2 + 2ax}{x} = \frac{a^2+x^2}{x}$,

and $\frac{(a+x)^2}{2ax} - 1 = \frac{a^2 + 2ax + x^2 - 2ax}{2ax} = \frac{a^2+x^2}{2ax}$;

$\therefore$ the quotient $= \frac{a^2+x^2}{x} \div \frac{a^2+x^2}{2ax} = \frac{a^2+x^2}{x} \times \frac{2ax}{a^2+x^2} = 2a$.

Ex. 4. Divide $\frac{8ab}{9x^2} + 2 + \frac{9x^2}{8ab}$ by $\frac{4a}{3x} + \frac{3x}{2b}$.

Here, proceeding as in integral quantities, we have the operation as follows:

$$\frac{4a}{3x} + \frac{3x}{2b}\Big)\ \frac{8ab}{9x^2} + 2 + \frac{9x^2}{8ab}\ \Big(\frac{2b}{3x} + \frac{3x}{4a} = \text{the quotient.}$$

$$\frac{8ab}{9x^2} + 1$$

$$\overline{\qquad\qquad 1 + \frac{9x^2}{8ab}}$$

$$1 + \frac{9x^2}{8ab}$$

In this example the fractions might have been transformed as in Ex. 4. of (75), and the quotient obtained would after reduction have been found to be the same.

80. Cor. 1. Since it is obvious that the quotient of

$\frac{ab}{c}$ by $b = \frac{ab}{c} \div \frac{b}{1} = \frac{ab}{c} \times \frac{1}{b} = \frac{ab}{bc} = \frac{a}{c}$;

we conclude that when a fraction is to be divided by an integral quantity, it is immaterial whether the numerator be divided by it, or the denominator be multiplied by it.

81. Cor. 2. A complex fraction as $\dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}$ is easily reduced to a simple one, for it manifestly $= \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

82. Cor. 3. As in (78), since $a^{-m}$ and $a^{-n}$ are equivalent to $\frac{1}{a^m}$ and $\frac{1}{a^n}$ respectively, we have

$$a^{-m} \div a^{-n} = \frac{1}{a^m} \div \frac{1}{a^n} = \frac{1}{a^m} \times a^n = \frac{a^n}{a^m} = a^{n-m}, \text{ or } \frac{1}{a^{m-n}}:$$

also $a^m \div a^{-n} = a^m \div \frac{1}{a^n} = a^m \times a^n = a^{m+n}$, or $\frac{1}{a^{-m-n}}$;

so that Rule 2. of Article (21) may be made use of, whether the indices are positive or negative, or partly both.

## VI. INVOLUTION.

83. *To find the Powers of a Fraction.*

Let $\frac{a}{b}$ be any proposed fraction, then by (75) we shall have

$$\text{the square} = \frac{a}{b} \times \frac{a}{b} = \frac{a \times a}{b \times b} = \frac{a^2}{b^2};$$

$$\text{the cube} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a \times a \times a}{b \times b \times b} = \frac{a^3}{b^3}; \text{ \&c.};$$

$$\text{the } m^{\text{th}} \text{ power} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \text{\&c. to } m \text{ factors}$$

$$= \frac{a \times a \times a \times \text{\&c. to } m \text{ factors}}{b \times b \times b \times \text{\&c. to } m \text{ factors}} = \frac{a^m}{b^m}:$$

whence we infer that any power of a fraction is found by multiplying the indices of the numerator and denominator by that of the proposed power: or, which is the same thing, by raising both the numerator and denominator to the power required.

Ex. 1. The square of $-\frac{ax}{b}$ is $\frac{a^2x^2}{b^2}$; the cube is $-\frac{a^3x^3}{b^3}$; the fourth power is $\frac{a^4x^4}{b^4}$, &c.; the $m^{\text{th}}$ power is $\pm\frac{a^m x^m}{b^m}$.

Ex. 2. The square of $\frac{x+2y}{x-y}$ is $\frac{x^2+4xy+4y^2}{x^2-2xy+y^2}$;

the cube is $\frac{x^3+6x^2y+12xy^2+8y^3}{x^3-3x^2y+3xy^2-y^3}$; and so on.

Ex. 3. Find the square, cube, &c. of $ax - \frac{abx}{b+x}$.

First, we have $ax - \frac{abx}{b+x} = \frac{abx+ax^2-abx}{b+x} = \frac{ax^2}{b+x}$;

$\therefore$ the square $= \frac{a^2x^4}{(b+x)^2}$; the cube $= \frac{a^3x^6}{(b+x)^3}$; &c. = &c.

Ex. 4. Required the square, cube, &c. of $\frac{x}{a}+\frac{b}{x}$.

Here the root $= \frac{x}{a}+\frac{b}{x}$

$$\begin{array}{l}
\frac{x}{a}+\frac{b}{x} \\
\hline
\frac{x^2}{a^2}+\frac{b}{a} \\
\qquad\;\; \frac{b}{a}+\frac{b^2}{x^2} \\
\hline
\end{array}$$

$$\therefore \text{ the square} = \frac{x^2}{a^2} + \frac{2b}{a} + \frac{b^2}{x^2}.$$

$$\frac{x}{a} + \frac{b}{x}$$

$$\frac{x^3}{a^3} + \frac{2bx}{a^2} + \frac{b^2}{ax}$$

$$\frac{bx}{a^2} + \frac{2b^2}{ax} + \frac{b^3}{x^3}$$

$$\therefore \text{ the cube} = \frac{x^3}{a^3} + \frac{3bx}{a^2} + \frac{3b^2}{ax} + \frac{b^3}{x^3}; \text{ \&c.}$$

These are in fact the same as the square, cube, &c. of $\frac{x^2+ab}{ax}$ when expressed in the least terms.

84. Cor. Because $a^{-n}$ is equivalent to $\frac{1}{a^n}$, we shall manifestly have the $m^{\text{th}}$ power of $a^{-n} = \left(\frac{1}{a^n}\right)^m = \frac{1}{a^{mn}}$: also the $-m^{\text{th}}$ power of $a^n$ is equivalent to $(a^n)^{-m} = \frac{1}{(a^n)^m} = \frac{1}{a^{mn}} = a^{-mn}$: whence we conclude that Rule 1. of Article (22) extends to all cases whether the indices be positive or negative, or partly both.

## VII. EVOLUTION.

85. *To find the Roots of a Fraction.*

Since by (83) we have seen that $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, it follows that, by the reverse operation, the $m^{\text{th}}$ root of $\frac{a^m}{b^m}$ is $\frac{a}{b}$:

whence it is evident that any root of a fraction is obtained by extracting the required root of the numerator and denominator respectively.

Ex. 1. The square root of $\frac{a^6 x^{12}}{b^{18}} = \pm \frac{a^{\frac{6}{2}} x^{\frac{12}{2}}}{b^{\frac{18}{2}}} = \pm \frac{a^3 x^6}{b^9}$;

and the cube root $= \frac{a^{\frac{6}{3}} x^{\frac{12}{3}}}{b^{\frac{18}{3}}} = \frac{a^2 x^4}{b^6}$.

Ex. 2. The cube root of $-\frac{(a+2xy)^6}{(b-3cy)^3}$ is $-\frac{(a+2xy)^2}{b-3cy}$.

Ex. 3. Find the square root of $\frac{a^4+x^4}{x^2} + 2a^2$.

First, we have $\frac{a^4+x^4}{x^2} + 2a^2 = \frac{a^4+x^4+2a^2x^2}{x^2} = \frac{(a^2+x^2)^2}{x^2}$;

therefore the required square root $= \pm \frac{a^2+x^2}{x}$.

Ex. 4. Required the square root of $\frac{a^2}{b^2} - \frac{4a}{3c} + \frac{4b^2}{9c^2}$.

The terms being already arranged according to the dimensions of $a$, we have as in integral quantities,

$$
\begin{array}{rl}
 & \frac{a^2}{b^2} - \frac{4a}{3c} + \frac{4b^2}{9c^2}\left(\frac{a}{b} - \frac{2b}{3c} = \text{the square root.}\right. \\
 & \frac{a^2}{b^2} \\
\hline
\left.\frac{2a}{b} - \frac{2b}{3c}\right) & - \frac{4a}{3c} + \frac{4b^2}{9c^2} \\
 & - \frac{4a}{3c} + \frac{4b^2}{9c^2} \\
\hline
\end{array}
$$

In this example, we might have reduced the parts of the quantity proposed to a common denominator, and then

have extracted the root of the numerator and denominator separately.

86. Cor. Since $a^{-mn}$ is equivalent to $\frac{1}{a^{mn}}$, it is obvious that the $m^{\text{th}}$ root of $a^{-mn}$ is $\frac{1}{a^n} = a^{-n}$: also, the $-m^{\text{th}}$ root of $a^{-mn}$, which is equivalent to the $m^{\text{th}}$ root of $\frac{1}{a^{-mn}}$, or $a^{mn}$ is $a^n$: hence Rule 1. of Article (23) extends to all cases, whether the indices be positive or negative, or partly both.

87. In Article (69) it has been seen that a fraction may sometimes be expressed by a mixed quantity, but it frequently happens that by the operation of division, a fraction may be expressed by means of a series of integral quantities, or of other fractions which never terminates; in other words, that a fraction may be converted into what is called an infinite series. This will be best illustrated by Examples.

Ex. 1. Let the proposed fraction be $\frac{a}{1-b}$: then by actual division we have

$$
\begin{array}{l}
1-b)\ a \qquad\qquad (a+ab+ab^2+\&c. \\
\quad \underline{a-ab} \\
\qquad ab \\
\qquad \underline{ab-ab^2} \\
\qquad\qquad ab^2 \\
\qquad\qquad \underline{ab^2-ab^3} \\
\qquad\qquad\qquad ab^3,
\end{array}
$$

in which it is obvious that the operation may be continued as long as we please, so that the entire operation would be endless, and the quotient an infinite series $a+ab+ab^2+$ &c.

Ex. 2. Let $\frac{a}{b+c}$ be the fraction proposed, then we have

$$\frac{a}{b+c} = \frac{a}{b} - \frac{ac}{b(b+c)} = \frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^2(b+c)}$$

$$= \frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^3} - \frac{ac^3}{b^3(b+c)}$$

$$= \frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^3} - \frac{ac^3}{b^4} + \text{\&c.} \pm \frac{ac^n}{b^n(b+c)}.$$

In these cases, the law according to which the successive terms are formed is manifest, and a few steps only are necessary to point it out.

88. In the examples just given, it is usual to write

$$\frac{a}{1-b} = a + ab + ab^2 + \text{\&c. } \textit{in infinitum:}$$

$$\frac{a}{b+c} = \frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^3} - \text{\&c. } \textit{in infinitum:}$$

but it may be observed that the symbol = then no longer denotes arithmetical equality between the quantities on each side of it, but merely indicates that the former quantity may be exhibited in the latter form: and in what is called the expansion or developement of algebraical quantities, the symbol has generally this particular signification. If, however, at any step of the operation the remainder be retained, the symbol keeps its proper signification, and the equation will be numerically exact, as

$$\frac{a}{1-b} = a + ab + ab^2 + \frac{ab^3}{1-b},$$

$$\frac{a}{b+c} = \frac{a}{b} - \frac{ac}{b^2} + \frac{ac^2}{b^3} - \frac{ac^3}{b^3(b+c)}.$$

89. Cor. 1. It having been seen that by an extension of the meaning of the symbol = we may have

$$\frac{a}{1-b} \doteq a + ab + ab^2 + \&c. \textit{ in infinitum:}$$

if we suppose $b = 1$, we shall, on the same supposition, have

$$\frac{a}{0} = a + a + a + \&c. \textit{ in infinitum,}$$

which is indefinitely great: and hence, according to this extended signification of the symbol =, if infinite numerical magnitude be denoted by the symbol $\infty$, we shall manifestly have

$$\frac{a}{0} = \infty, \quad \therefore \quad \frac{a}{\infty} = 0 \text{ and } 0 \times \infty = a.$$

These results may be enunciated generally, in the following terms:

A finite quantity divided by zero gives an infinite quotient: a finite quantity divided by infinity gives zero for a quotient, and the product of zero and infinity may be finite.

90. Cor. 2. It moreover sometimes happens, that by assigning a particular value to one of the quantities involved in the terms of a fraction, the result appears under the indeterminate form $\frac{0}{0}$, from which it is obvious that no definite conclusion can be immediately derived. This peculiarity being the consequence of some factor involved in the numerator and denominator becoming equal to 0, may manifestly be removed by dividing both of them by such factor determined according to the rule laid down in (55).

Quantities exhibiting this peculiarity in their forms are termed *Vanishing Fractions.*

Ex. 1. Let it be required to find the value of the fraction $\frac{x^4 - a^4}{x^3 - a^3}$ in the particular case when $x = a$.

The fraction here becomes $\frac{a^4 - a^4}{a^3 - a^3}$ or $\frac{0}{0}$, but it is readily discovered that the common factor which occasions this par-

ticular form is $x-a$, and therefore if the numerator and denominator be both divided by it we shall have the fraction $=\dfrac{x^3+ax^2+a^2x+a^3}{x^2+ax+a^2}$ whose value, when $x=a$, is manifestly

$$=\frac{4a^3}{3a^2}=\frac{4}{3}a.$$

Ex. 2. If in the fraction $\dfrac{8x^3-4x^2-2x+1}{12x^3+4x^2-3x-1}$ we suppose $x=\frac{1}{2}$ or $-\frac{1}{2}$, the numerator and denominator both become evanescent, but the fraction being reduced to its lowest terms is $\dfrac{2x-1}{3x+1}$ whose corresponding values are 0 and 4.

91. Having now seen the application of all the fundamental rules of arithmetic to algebraical fractions, we shall conclude this chapter with a few deductions from the principles which have been laid down and explained in it.

92. *If we have* $\dfrac{a}{b}=\dfrac{c}{d}$, *then will* $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

For, since $\dfrac{a}{b}=\dfrac{c}{d}$, we have $\dfrac{a}{b}\pm 1=\dfrac{c}{d}\pm 1$,

that is, $\dfrac{a+b}{b}=\dfrac{c+d}{d}$ and $\dfrac{a-b}{b}=\dfrac{c-d}{d}$:

whence $\dfrac{a+b}{b}\div\dfrac{a-b}{b}=\dfrac{c+d}{d}\div\dfrac{c-d}{d}$;

$\therefore\ \dfrac{a+b}{b}\times\dfrac{b}{a-b}=\dfrac{c+d}{d}\times\dfrac{d}{c-d}$, or $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$.

Hence also, conversely, if $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$, we shall have $\dfrac{a}{b}=\dfrac{c}{d}$.

93. *Again if* $\dfrac{a}{b}=\dfrac{c}{d}$, *then will* $\dfrac{a^m+b^m}{a^m-b^m}=\dfrac{c^m+d^m}{c^m-d^m}$.

For, since $\left(\frac{a}{b}\right)^m = \left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m} = \frac{c^m}{d^m}$, we have, by the last article, $\frac{a^m + b^m}{a^m - b^m} = \frac{c^m + d^m}{c^m - d^m}$: and hence also the converse is true.

94. *If* $\frac{\mathrm{a}}{\mathrm{b}}$ *be any fraction whatever, then will* $\frac{\mathrm{a}}{\mathrm{b}} + \frac{\mathrm{b}}{\mathrm{a}}$ *be always greater than* 2.

For, since by (22) it is proved that $(a - b)^2$ is always a positive quantity, whether $a$ be greater or less than $b$, we have

$$a^2 - 2ab + b^2 > 0,$$

and $\therefore$ by adding $2ab$ to both sides, we conclude that

$$a^2 + b^2 > 0 + 2ab > 2ab;$$

whence $\frac{a^2 + b^2}{ab}$ or $\frac{a}{b} + \frac{b}{a}$ is greater than 2: that is, the sum of any quantity (except unity) and its reciprocal is greater than 2.

95. *On the same hypothesis* $\frac{\mathrm{a}}{\mathrm{b}^2} + \frac{\mathrm{b}}{\mathrm{a}^2}$ *is greater than* $\frac{1}{\mathrm{b}} + \frac{1}{\mathrm{a}}$.

For, from the last article, $a^2 + b^2$ is $> 2ab$,

$$\therefore\ a^2 - ab + b^2 \text{ is } > ab;$$

and multiplying both sides by $a + b$, we have

$$a^3 + b^3 > a^2b + ab^2,$$

whence $\frac{a^3 + b^3}{a^2b^2}$ is $> \frac{a^2b + ab^2}{a^2b^2}$, or $\frac{a}{b^2} + \frac{b}{a^2}$ is $> \frac{1}{b} + \frac{1}{a}$.

Similarly, $\frac{a}{b^2} - \frac{b}{a^2}$ is proved to be greater than $\frac{1}{b} - \frac{1}{a}$.

96. If we have an equation, as $\frac{a}{b} = \frac{c}{d}$, and both sides be multiplied by $bd$, then $\frac{a}{b} \times bd = \frac{c}{d} \times bd$, or $ad = bc$.

Whence we infer that an equation may be *cleared of Fractions* by multiplying all its terms by the product of their denominators, or by their least common multiple.

Ex. 1. Clear of fractions the equation $\frac{ax}{b+x} - \frac{bx}{a+x} = c$.

Here, multiplying every term of both sides of the equation by $(b+x)\ (a+x)$, we have

$$ax\,(a+x) - bx\,(b+x) = c\,(a+x)\ (b+x),$$

$$\text{or}\ \ a^2x + ax^2 - b^2x - bx^2 = abc + (a+b)\,cx + cx^2,$$

$$\text{or}\ \ (a-b)\,x^2 + (a^2-b^2)\,x = abc + (a+b)\,cx + cx^2,$$

which is cleared of fractions: and by (44) we have further

$$(a-b-c)\,x^2 + \{a^2 - (a+b)\,c - b^2\}\,x = abc.$$

Ex. 2. Let the proposed equation be

$$\frac{a}{a-x} - \frac{ax}{a^2-x^2} + \frac{ax^2}{a^3-x^3} = 0.$$

Here the least common multiple of the denominators determined by (63) is $a^4 + a^3x - ax^3 - x^4$, and multiplying every term of the equation by this quantity, we have

$$a^4 + 2\,a^3x + 2\,a^2x^2 + ax^3 - a^3x - a^2x^2 - ax^3 + a^2x^2 + ax^3 = 0,$$

$$\text{or}\ \ ax^3 + 2\,a^2x^2 + a^3x + a^4 = 0,$$

by arranging the quantities according to the dimensions of $x$: and the equation may be still further simplified by dividing every term by $a$, so that there results the equation

$$x^3 + 2\,ax^2 + a^2x + a^3 = 0.$$

# CHAP. V.

## *On Irrational or Surd, and Imaginary or Impossible Quantities.*

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97. Def. An *Irrational* or *Surd* algebraical quantity is one in which the root indicated or expressed cannot be exactly extracted, and is generally characterized by means of the radical sign, or by a fraction as its index.

Thus,

$$\sqrt{a},\ \sqrt[3]{(a+x)^2},\ \sqrt[m]{(a^2-bx)^n},\ \sqrt{\frac{ax}{by}}\ \text{and}\ \frac{\sqrt{a^2-x^2}}{\sqrt[4]{2by-y^2}},$$

are all algebraical surds or irrational quantities: and these, by an extension of the notation explained in Definition (9) may be likewise conveniently written in the following forms;

$$a^{\frac{1}{2}},\ (a+x)^{\frac{2}{3}},\ (a^2-bx)^{\frac{n}{m}},\ \left(\frac{ax}{by}\right)^{\frac{1}{2}}\ \text{and}\ \frac{(a^2-x^2)^{\frac{1}{2}}}{(2by-y^2)^{\frac{1}{4}}},$$

wherein the numerators of the indices denote the powers to which the quantities are to be raised, and the denominators the roots intended to be extracted therefrom.

### I. REDUCTION.

98. *To represent a Rational Quantity as a Surd.*

Let $a$ be any rational quantity; then since unity may be considered as its index, we have $a = a^1 = a^{\frac{2}{2}} = a^{\frac{3}{3}} =$ &c. $= a^{\frac{m}{m}}$; and these latter may be also written $\sqrt{a^2}$, $\sqrt[3]{a^3}$, &c., $\sqrt[m]{a^m}$.

99. Cor. 1. Agreeably to the convention made in the latter part of Definition (10), the quantities $\sqrt{a^2}$, $\sqrt[3]{a^3}$, &c., $\sqrt[m]{a^m}$ may be written

$$\sqrt{\frac{1}{a^{-2}}},\ \sqrt[3]{\frac{1}{a^{-3}}},\ \&c.,\ \sqrt[m]{\frac{1}{a^{-m}}},$$

$$\text{or } \left(\frac{1}{a^{-2}}\right)^{\frac{1}{2}},\ \left(\frac{1}{a^{-3}}\right)^{\frac{1}{3}},\ \&c.,\ \left(\frac{1}{a^{-m}}\right)^{\frac{1}{m}}.$$

100. Cor. 2. Hence also the converse is true that a quantity which appears under the form of a surd may really be a rational quantity. Thus,

$$\sqrt{a^2+2ax+x^2}=a+x,\quad \sqrt{\frac{a^3-3a^2+3a-1}{a-1}}$$

$$\sqrt{a^2-2a+1}=a-1,\quad \sqrt[3]{\frac{a^3b^3x^3}{(a+b+x)^3}}=\frac{abx}{a+b+x}.$$

101. *To represent a Mixed Quantity as a Surd.*

Let $a\sqrt[m]{b}$ be the proposed quantity which is partly rational and partly surd; then since $a\sqrt[m]{b}=ab^{\frac{1}{m}}$ we have

$$(a\sqrt[m]{b})^m=(ab^{\frac{1}{m}})^m=a^mb,$$

whence immediately results $a\sqrt[m]{b}=\sqrt[m]{a^mb}=(a^mb)^{\frac{1}{m}}$.

Ex. 1. Let $xy^{\frac{1}{2}}$ be the proposed mixed quantity; then since $x=x^{\frac{1}{2}\times 2}$, we shall manifestly have

$$xy^{\frac{1}{2}}=x^{\frac{1}{2}\times 2}y^{\frac{1}{2}}=(x^2y)^{\frac{1}{2}}=\sqrt{x^2y}.$$

Ex. 2. Let the proposed quantity be $\dfrac{x+1}{x-1}\sqrt{\dfrac{x-1}{x+1}}$;

then since $\left(\dfrac{x+1}{x-1}\right)^2=\left(\dfrac{x+1}{x-1}\right)^2$ and $\therefore\ \dfrac{x+1}{x-1}=\sqrt{\left(\dfrac{x+1}{x-1}\right)^2}$,

we shall have

$$\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}} = \sqrt{\left(\frac{x+1}{x-1}\right)^2\left(\frac{x-1}{x+1}\right)} = \sqrt{\frac{x+1}{x-1}}.$$

102. Cor. Conversely, a surd may frequently be represented by a mixed quantity, for $\sqrt[m]{a^m b}$ is obviously the same as $a\sqrt[m]{b}$.

Ex. 1. The surd $\sqrt{(a+bx)^4 xy}$ is equivalent to the mixed quantity $(a+bx)^2\sqrt{xy} = (a^2+2abx+b^2x^2)\sqrt{xy}$.

Ex. 2. If the surd proposed be $\sqrt[3]{(a+x)^3 b^2}$, it is obviously equivalent to $(a+x)\sqrt[3]{b^2}$, or $(a+x)\,b^{\frac{2}{3}}$.

Ex. 3. Let the proposed surd be $\sqrt[m]{\dfrac{(cx-x^2)^{2m}(a+x)}{b+x}}$;

then this may manifestly be written $\sqrt[m]{(cx-x^2)^{2m}}\sqrt[m]{\dfrac{a+x}{b+x}}$

which is equal to

$$(cx-x^2)^2\sqrt[m]{\frac{a+x}{b+x}},\ \text{or}\ (cx-x^2)^2\left(\frac{a+x}{b+x}\right)^{\frac{1}{m}}.$$

103. *To reduce a Surd to its simplest Form.*

Since $\sqrt[m]{a^{m+n}b}$ is equivalent to

$$\sqrt[m]{a^m \times a^n b} = \sqrt[m]{a^m}\sqrt[m]{a^n b} = a\sqrt[m]{a^n b};$$

it follows that a surd may be reduced to its simplest form by seeking in the expression affected by the radical sign or fractional index, the greatest factor of which the root expressed can be extracted, and retaining the rest with the proper index.

Ex. 1. Let the proposed surd be $\sqrt{27a^3x^5}$.

Here $\sqrt{27\,a^3x^5} = \sqrt{9\,a^2x^4 \times 3\,ax} = \sqrt{9\,a^2x^4}\,\sqrt{3\,ax}$
$= 3\,ax^2\,\sqrt{3\,ax}.$

Ex. 2. If the surd be $\sqrt[3]{5\,ax^4 - 3\,b^2x^3}$, we shall have it in its simplest form $= \sqrt[3]{x^3 \times (5\,ax - 3\,b^2)} = x\,\sqrt[3]{5\,ax - 3\,b^2}$.

Ex. 3. The surd $\sqrt{\dfrac{ax^2}{a-x}}$ reduced to its simplest form

becomes $= \sqrt{\dfrac{ax^2\,(a-x)}{(a-x)^2}} = \sqrt{\dfrac{x^2}{(a-x)^2}}\,\sqrt{a\,(a-x)}$

$$= \frac{x}{a-x}\,\sqrt{a\,(a-x)}.$$

104. *To reduce Surds to others having a common Index.*

Let $\sqrt[m]{a}$ and $\sqrt[n]{b}$, which are equivalent to $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$, be the surds proposed; then it is manifest that by reducing their indices $\frac{1}{m}$ and $\frac{1}{n}$ to others having a common denominator as $\frac{n}{mn}$ and $\frac{m}{mn}$, these quantities become $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$, which may be also written $(a^n)^{\frac{1}{mn}}$ and $(b^m)^{\frac{1}{mn}}$, or $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$, having thus the common index $\frac{1}{mn}$.

Ex. 1. Reduce $\sqrt{ax}$ and $\sqrt[3]{bx^2}$ to surds having a common index.

Here $\sqrt{ax} = a^{\frac{1}{2}}x^{\frac{1}{2}} = a^{\frac{3}{6}}x^{\frac{3}{6}} = (a^3x^3)^{\frac{1}{6}} = \sqrt[6]{a^3x^3}$,

and $\sqrt[3]{bx^2} = b^{\frac{1}{3}}x^{\frac{2}{3}} = b^{\frac{2}{6}}x^{\frac{4}{6}} = (b^2x^4)^{\frac{1}{6}} = \sqrt[6]{b^2x^4}$,

so that the new equivalent surds are $(a^3x^3)^{\frac{1}{6}}$ and $(b^2x^4)^{\frac{1}{6}}$ having the common index $\frac{1}{6}$.

Ex. 2. Let the quantities proposed be $a\sqrt{a-x}$ and $b\sqrt[3]{a^2-x^2}$; then $a\sqrt{a-x}=\sqrt{a^3-a^2x}=(a^3-a^2x)^{\frac{1}{2}}$

$=(a^3-a^2x)^{\frac{3}{6}}=\{(a^3-a^2x)^3\}^{\frac{1}{6}}=(a^9-3a^8x+3a^7x^2-a^6x^3)^{\frac{1}{6}}$;

and $b\sqrt[3]{a^2-x^2}=\sqrt[3]{a^2b^3-b^3x^2}=(a^2b^3-b^3x^2)^{\frac{1}{3}}$

$=(a^2b^3-b^3x^2)^{\frac{2}{6}}=\{(a^2b^3-b^3x^2)^2\}^{\frac{1}{6}}=(a^4b^6-2a^2b^6x^2+b^6x^4)^{\frac{1}{6}}$:

whence the required surds are

$$\sqrt[6]{a^9-3a^8x+3a^7x^2-a^6x^3} \text{ and } \sqrt[6]{a^4b^6-2a^2b^6x^2+b^6x^4}.$$

Ex. 3. The surds $a\sqrt{x-y}$ and $\dfrac{b}{\sqrt[4]{x+y}}$ are by (101) equivalent to $(a^2x-a^2y)^{\frac{1}{2}}$ and $\left(\dfrac{b^4}{x+y}\right)^{\frac{1}{4}}$ respectively; whence the new surds having the common index $\frac{1}{4}$ will manifestly be $\{(a^2x-a^2y)^2\}^{\frac{1}{4}}$ and $\left(\dfrac{b^4}{x+y}\right)^{\frac{1}{4}}$, or $(a^4x^2-2a^4xy+a^4y^2)^{\frac{1}{4}}$ and $\{b^4(x+y)^{-1}\}^{\frac{1}{4}}$.

105. Cor. If the common index be given, the indices of the proposed surds must manifestly be reduced to fractions having the same denominator with it.

Ex. If it be required to reduce $\sqrt{a^2-x^2}$ and $\sqrt[4]{a^4+x^4}$ to others with the common index $\frac{1}{8}$, we have

$$\sqrt{a^2-x^2}=(a^2-x^2)^{\frac{1}{2}}=(a^2-x^2)^{\frac{4}{8}}=\{(a^2-x^2)^4\}^{\frac{1}{8}}$$

$=(a^8-4a^6x^2+6a^4x^4-4a^2x^6+x^8)^{\frac{1}{8}}$, and $\sqrt[4]{a^4+x^4}=(a^4+x^4)^{\frac{1}{4}}$

$=(a^4+x^4)^{\frac{2}{8}}=\{(a^4+x^4)^2\}^{\frac{1}{8}}=(a^8+2a^4x^4+x^8)^{\frac{1}{8}}$.

## II. ADDITION.

106. *To find the Sum of two Surds.*

Let $\sqrt[m]{a^m b}$ and $\sqrt[m]{b x^{2m}}$ be the two proposed surds; then, by reduction to their simplest forms, we have

$$\sqrt[m]{a^m b} = \sqrt[m]{a^m}\sqrt[m]{b} = a\sqrt[m]{b},$$

$$\text{and } \sqrt[m]{b x^{2m}} = \sqrt[m]{x^{2m}}\sqrt[m]{b} = x^2\sqrt[m]{b};$$

$$\text{whence their sum} = a\sqrt[m]{b} + x^2\sqrt[m]{b} = (a + x^2)\sqrt[m]{b}.$$

If the two surds cannot be reduced to others having the same irrational factor, the addition can be indicated only by means of the proper sign. Similarly if there be more than two.

Ex. 1. Add together $\sqrt{4ax^2}$ and $3x\sqrt{9a}$.

$$\text{Here } \sqrt{4ax^2} = \sqrt{4x^2 \times a} = \sqrt{4x^2}\sqrt{a} = 2x\sqrt{a},$$

$$\text{and } 3x\sqrt{9a} = 3x\sqrt{9 \times a} = 3x\sqrt{9}\sqrt{a} = 9x\sqrt{a};$$

$$\therefore \text{ the required sum} = (2x + 9x)\sqrt{a} = 11x\sqrt{a}.$$

Ex. 2. Find the sum of the irrational quantities

$$3x\sqrt[3]{2a^5x^2},\ 8a\sqrt[3]{2a^2x^5} \text{ and } 2ax\sqrt[3]{2a^2x^2}.$$

$$\text{Here } 3x\sqrt[3]{2a^5x^2} = 3x\sqrt[3]{a^3 \times 2a^2x^2} = 3ax\sqrt[3]{2a^2x^2},$$

$$8a\sqrt[3]{2a^2x^5} = 8a\sqrt[3]{x^3 \times 2a^2x^2} = 8ax\sqrt[3]{2a^2x^2},$$

$$\text{and } 2ax\sqrt[3]{2a^2x^2} = 2ax\sqrt[3]{2a^2x^2};$$

whence we shall manifestly have the required sum

$$= (3ax + 8ax + 2ax)\sqrt[3]{2a^2x^2} = 13ax\sqrt[3]{2a^2x^2}.$$

Ex. 3. Required the sum of the fractional surds

$$\sqrt{\frac{a^4x}{b^3}},\ \sqrt{\frac{a^2x^3}{bc^2}} \text{ and } \sqrt{\frac{a^2c^2x}{bd^2}}.$$

Here $\sqrt{\frac{a^4x}{b^3}}=\sqrt{\frac{a^4}{b^2}\times\frac{x}{b}}=\sqrt{\frac{a^4}{b^2}}\sqrt{\frac{x}{b}}=\frac{a^2}{b}\sqrt{\frac{x}{b}},$

$$\sqrt{\frac{a^2x^3}{bc^2}}=\sqrt{\frac{a^2x^2}{c^2}\times\frac{x}{b}}=\sqrt{\frac{a^2x^2}{c^2}}\sqrt{\frac{x}{b}}=\frac{ax}{c}\sqrt{\frac{x}{b}},$$

$$\sqrt{\frac{a^2c^2x}{bd^2}}=\sqrt{\frac{a^2c^2}{d^2}\times\frac{x}{b}}=\sqrt{\frac{a^2c^2}{d^2}}\sqrt{\frac{x}{b}}=\frac{ac}{d}\sqrt{\frac{x}{b}};$$

whence the required sum $=\left(\frac{a^2}{b}+\frac{ax}{c}+\frac{ac}{d}\right)\sqrt{\frac{x}{b}}.$

Ex. 4. Required the sum of the surds

$$\sqrt{\frac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}} \text{ and } \sqrt{\frac{a^2x+2ax^2+x^3}{a^2-2ax+x^2}}.$$

In this case we have by reduction,

$$\sqrt{\frac{a^2x-2ax^2+x^3}{a^2+2ax+x^2}}=\sqrt{\frac{a^2-2ax+x^2}{a^2+2ax+x^2}}\sqrt{x}=\frac{a-x}{a+x}\sqrt{x},$$

$$\sqrt{\frac{a^2x+2ax^2+x^3}{a^2-2ax+x^2}}=\sqrt{\frac{a^2+2ax+x^2}{a^2-2ax+x^2}}\sqrt{x}=\frac{a+x}{a-x}\sqrt{x};$$

whence the sum required

$$=\left(\frac{a-x}{a+x}+\frac{a+x}{a-x}\right)\sqrt{x}=2\left(\frac{a^2+x^2}{a^2-x^2}\right)\sqrt{x}.$$

## III. SUBTRACTION.

107. *To find the Difference of two Surds.*

Let $\sqrt[m]{a^m b}$ and $\sqrt[m]{b x^{2m}}$ be the surds proposed, then as in the last article these are equivalent to $a\sqrt[m]{b}$ and $x^2\sqrt[m]{b}$ respectively; whence their difference is obviously $(a-x^2)\sqrt[m]{b}$.

Also, if they be not reducible so as to have the same irrational factor, their difference can be expressed only by means of the proper algebraical sign.

Ex. 1. From $9a\sqrt{bx^2}$ take $5x\sqrt{a^2b}$.

Here $9a\sqrt{bx^2} = 9ax\sqrt{b}$, and $5x\sqrt{a^2b} = 5ax\sqrt{b}$; whence the remainder required $= (9ax - 5ax)\sqrt{b} = 4ax\sqrt{b}$.

Ex. 2. Subtract $a\sqrt{bc^2}$ from $\sqrt[4]{16a^4b^2c^4}$.

Here $a\sqrt{bc^2} = ac\sqrt{b}$, and $\sqrt[4]{16a^4b^2c^4} = 2ac\sqrt{b}$; therefore the remainder $= (2ac - ac)\sqrt{b} = ac\sqrt{b}$.

Ex. 3. From $\sqrt[3]{\dfrac{27a^4x^4}{2b}}$ take $\sqrt[3]{\dfrac{ax^4}{54b}}$.

Here $\sqrt[3]{\dfrac{27a^4x^4}{2b}} = 3ax\sqrt[3]{\dfrac{ax}{2b}}$, and $\sqrt[3]{\dfrac{ax^4}{54b}} = \dfrac{x}{3}\sqrt[3]{\dfrac{ax}{2b}}$;

whence the required remainder $= \left(3ax - \dfrac{x}{3}\right)\sqrt[3]{\dfrac{ax}{2b}}$.

Ex. 4. The difference of the surds $\sqrt{\dfrac{a^2b - 2ab^2 + b^3}{a^2 + 2ab + b^2}}$

and $\sqrt{\dfrac{a^2b + 2ab^2 + b^3}{a^2 - 2ab + b^2}} = \left(\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}\right)\sqrt{b} = \dfrac{4ab\sqrt{b}}{a^2 - b^2}$.

## IV. MULTIPLICATION.

108. *To find the Product of two Surds.*

Let $\sqrt[m]{a}$ and $\sqrt[n]{b}$ be any two surds; then it is manifest that their product

$$= \sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n} \times \sqrt[mn]{b^m} = \sqrt[mn]{a^n b^m} = (a^n b^m)^{\frac{1}{mn}}.$$

If rational quantities be involved in one or both of the factors, their product must evidently be annexed to the product of the surds by means of the proper algebraical sign.

Similarly for the continued product of any number of surds whatever.

Ex. 1. The product of $\sqrt{a^3 x}$ and $\sqrt{ax(a^2 - x^2)}$

$$= \sqrt{a^3 x \times ax(a^2 - x^2)} = \sqrt{a^4 x^2 (a^2 - x^2)} = a^2 x \sqrt{a^2 - x^2}.$$

Ex. 2. Required the product of $a\sqrt[3]{x}$ and $b\sqrt[4]{y}$.

Here $a\sqrt[3]{x} = ax^{\frac{1}{3}} = ax^{\frac{4}{12}}$ and $b\sqrt[4]{y} = by^{\frac{1}{4}} = by^{\frac{3}{12}}$;

∴ we shall have the product required

$$= ax^{\frac{4}{12}} \times by^{\frac{3}{12}} = ab(x^4 y^3)^{\frac{1}{12}} = ab\sqrt[12]{x^4 y^3}.$$

Ex. 3. Find the continued product of the three surds

$$\frac{ax}{bc}\sqrt{ax}, \quad \frac{by}{cd}\sqrt[3]{by}, \quad \text{and} \quad \frac{c^2 d}{a}\sqrt[4]{cz}.$$

The required product will obviously be

$$= \frac{ax}{bc} \times \frac{by}{cd} \times \frac{c^2 d}{a}\sqrt{ax}\,\sqrt[3]{by}\,\sqrt[4]{cz}$$

$$= xy(ax)^{\frac{1}{2}}(by)^{\frac{1}{3}}(cz)^{\frac{1}{4}} = xy\,a^{\frac{6}{12}} x^{\frac{6}{12}} b^{\frac{4}{12}} y^{\frac{4}{12}} c^{\frac{3}{12}} z^{\frac{3}{12}}$$

$$= xy\sqrt[12]{a^6 b^4 c^3 x^6 y^4 z^3}.$$

Ex. 4. Multiply $ab + c\sqrt{xy}$ by $a - \sqrt{z}$.

Proceeding as in rational quantities, we have

$$\begin{array}{l} ab + c\sqrt{xy} \\ a - \sqrt{z} \\ \hline a^2b + ac\sqrt{xy} \\ \qquad - ab\sqrt{z} - c\sqrt{xyz} \\ \hline \end{array}$$

$\therefore$ the product $= a^2b + ac\sqrt{xy} - ab\sqrt{z} - c\sqrt{xyz}$.

109. Cor. If in the last article we suppose $b = a$, we shall have the product of $\sqrt[m]{a}$ and $\sqrt[n]{a} = \sqrt[mn]{a^n a^m} = \sqrt[mn]{a^{m+n}}$ $a^{\frac{m+n}{mn}} = a^{\frac{1}{m}+\frac{1}{n}}$: and hence Rule (2) laid down in Article (20), with respect to the multiplication of different powers of the same rational quantity, holds good also in the multiplication of quantities of all kinds, whether the indices be positive or negative, integral or fractional.

Ex. 1. Multiply $x - \sqrt{xy} + y$ by $\sqrt{x} + \sqrt{y}$.

Here, as in rational quantities, we have

$$\begin{array}{l} x - \sqrt{xy} + y \\ \sqrt{x} + \sqrt{y} \\ \hline x\sqrt{x} - x\sqrt{y} + y\sqrt{x} \\ \qquad + x\sqrt{y} - y\sqrt{x} + y\sqrt{y} \\ \hline \end{array}$$

$\therefore$ the product $= x\sqrt{x} + y\sqrt{y}$ or $x^{\frac{3}{2}} + y^{\frac{3}{2}}$.

Ex. 2. Proceeding as in integral quantities, we have

$$a^{\frac{5}{2}} + a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} + ab + a^{\frac{1}{2}} b^{\frac{4}{3}} + b^{\frac{5}{3}}$$

$$a^{\frac{1}{2}} - b^{\frac{1}{3}}$$

$$a^3 + a^{\frac{5}{2}} b^{\frac{1}{3}} + a^2 b^{\frac{2}{3}} + a^{\frac{3}{2}} b + ab^{\frac{4}{3}} + a^{\frac{1}{2}} b^{\frac{5}{3}}$$

$$- a^{\frac{5}{2}} b^{\frac{1}{3}} - a^2 b^{\frac{2}{3}} - a^{\frac{3}{2}} b - ab^{\frac{4}{3}} - a^{\frac{1}{2}} b^{\frac{5}{3}} - b^2$$

$\therefore$ the product $= a^3 - b^2$.

## V. DIVISION.

110. *To find the Quotient of two Surds.*

Taking $\sqrt[m]{a}$ and $\sqrt[n]{b}$ for any two surds, we shall obviously have the quotient resulting from the division of the former by the latter

$$= \frac{\sqrt[m]{a}}{\sqrt[n]{b}} = \frac{\sqrt[mn]{a^n}}{\sqrt[mn]{b^m}} = \sqrt[mn]{\frac{a^n}{b^m}} = \left(\frac{a^n}{b^m}\right)^{\frac{1}{mn}}.$$

If the surds have rational quantities connected with them, their quotient must evidently be annexed to that of the surds by means of the proper algebraical sign.

Ex. 1. The quotient of $\sqrt{ax^3}$ divided by $\sqrt{bx}$

$$= \sqrt{\frac{ax^3}{bx}} = \sqrt{\frac{ax^2}{b}} = x\sqrt{\frac{a}{b}} = \frac{x}{b}\sqrt{ab}.$$

Ex. 2. Find the quotient resulting from the division of $\sqrt{ax - x^2}$ by $\sqrt[3]{a^2 - x^2}$.

First, $\sqrt{ax-x^2} = (ax-x^2)^{\frac{1}{2}} = \{(ax-x^2)^3\}^{\frac{1}{6}} = \{x^3(a-x)^3\}^{\frac{1}{6}}$,

and

$\sqrt[3]{a^2-x^2} = (a^2-x^2)^{\frac{1}{3}} = \{(a^2-x^2)^2\}^{\frac{1}{6}} = \{(a+x)^2(a-x)^2\}^{\frac{1}{6}}$;

$\therefore$ the required quotient will obviously be

$$= \left(\frac{x^3(a-x)^3}{(a+x)^2(a-x)^2}\right)^{\frac{1}{6}} = \left\{\frac{x^3(a-x)}{(a+x)^2}\right\}^{\frac{1}{6}} = \frac{x^{\frac{1}{2}}(a-x)^{\frac{1}{6}}}{(a+x)^{\frac{1}{3}}}.$$

Ex. 3. What is the quotient arising from the division of

$$a\sqrt{\frac{bc-bx}{c}} \text{ by } b\sqrt{\frac{ad-ax}{d}}?$$

The required quotient will obviously be

$$= \frac{a}{b}\sqrt{\frac{bc-bx}{c}} \times \sqrt{\frac{d}{ad-ax}} = \frac{a}{b}\sqrt{\frac{b(c-x)d}{ca(d-x)}}$$

$$= \sqrt{\frac{a^2b(c-x)d}{b^2ca(d-x)}} = \sqrt{\frac{ad(c-x)}{bc(d-x)}}.$$

Ex. 4. Divide $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.

Here by the process used in rational quantities, we have

$$\begin{array}{l}
\sqrt{x}+\sqrt{y})\ a\sqrt{x}+a\sqrt{y}-\sqrt{bx}-\sqrt{by}\ (a-\sqrt{b}, \\
\qquad\qquad\ \underline{a\sqrt{x}+a\sqrt{y}\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad\quad -\sqrt{bx}-\sqrt{by} \\
\qquad\qquad\qquad\qquad\quad \underline{-\sqrt{bx}-\sqrt{by}}
\end{array}$$

so that the quotient is $a-\sqrt{b}$.

111. Cor. Since the quotient arising from the division of $\sqrt[m]{a}$ by $\sqrt[n]{b}$ has been shewn to be $\left(\frac{a^n}{b^m}\right)^{\frac{1}{mn}}$, if we make $b = a$, we shall have $\sqrt[m]{a} \div \sqrt[n]{a} = \left(\frac{a^n}{a^m}\right)^{\frac{1}{mn}} = a^{\frac{n-m}{mn}} = a^{\frac{1}{m}-\frac{1}{n}}$, which shews that Rule (2) of Article (21) holds equally for surds and rational quantities.

Ex. 1. Let $a + b - c + 2\sqrt{ab}$ be divided by

$$\sqrt{a} + \sqrt{b} - \sqrt{c}.$$

Proceeding according to the usual method, we have

$$\sqrt{a} + \sqrt{b} - \sqrt{c})\ a + 2\sqrt{ab} + b - c\ (\sqrt{a} + \sqrt{b} + \sqrt{c},$$

$$\underline{a + \sqrt{ab} - \sqrt{ac}}$$

$$\sqrt{ab} + b + \sqrt{ac}$$

$$\underline{\sqrt{ab} + b - \sqrt{bc}}$$

$$\sqrt{ac} + \sqrt{bc} - c$$

$$\underline{\sqrt{ac} + \sqrt{bc} - c}$$

whence the quotient is $\sqrt{a} + \sqrt{b} + \sqrt{c}$.

Ex. 2. Divide $a^3 - b^3$ by $a^{\frac{3}{4}} + b^{\frac{3}{4}}$.

Here, according to the rule, we have

$$a^{\frac{3}{4}} + b^{\frac{3}{4}})\ a^3 - b^3\ (a^{\frac{9}{4}} - a^{\frac{3}{2}}b^{\frac{3}{4}} + a^{\frac{3}{4}}b^{\frac{3}{2}} - b^{\frac{9}{4}} = \text{the quotient.}$$

$$\underline{a^3 + a^{\frac{9}{4}}b^{\frac{3}{4}}}$$

$$-a^{\frac{9}{4}}b^{\frac{3}{4}} - b^3$$

$$\underline{-a^{\frac{9}{4}}b^{\frac{3}{4}} - a^{\frac{3}{2}}b^{\frac{3}{2}}}$$

$$a^{\frac{3}{2}}b^{\frac{3}{2}} - b^3$$

$$a^{\frac{3}{2}}b^{\frac{3}{2}} + a^{\frac{3}{4}}b^{\frac{9}{4}}$$

$$- a^{\frac{3}{4}}b^{\frac{9}{4}} - b^3$$

$$- a^{\frac{3}{4}}b^{\frac{9}{4}} - b^3$$

## VI. INVOLUTION.

112. *To find the Powers of a Surd.*

Let $\sqrt[n]{a}$ be any proposed surd; then it is manifest that its $m^{\text{th}}$ power $= \sqrt[n]{a} \times \sqrt[n]{a} \times \sqrt[n]{a} \times$ &c. to $m$ factors $= \sqrt[n]{a \times a \times a \times \text{\&c. to } m \text{ factors}} = \sqrt[n]{a^m} = a^{\frac{m}{n}}$; that is, $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ raised to the $m^{\text{th}}$ power is $a^{\frac{m}{n}}$.

Whence any power of a surd is found by multiplying its index by that of the proposed power as in rational quantities.

Also, if any rational quantity be connected with the surd, it is obvious that its power must likewise be connected with that of the surd by the proper algebraical sign.

Ex. 1. The square of $\sqrt[4]{ax^3}$ is $(ax^3)^{\frac{1}{4}\times 2} = (ax^3)^{\frac{1}{2}} = \sqrt{ax^3}$; the cube is $(ax^3)^{\frac{1}{4}\times 3} = (ax^3)^{\frac{3}{4}} = \sqrt[4]{a^3x^9}$; &c.

Ex. 2. The cube of $\dfrac{\sqrt{a^2 - x^2}}{\sqrt{a}\sqrt[3]{a+x}}$ = the cube of $\dfrac{(a^2 - x^2)^{\frac{1}{2}}}{a^{\frac{1}{2}}(a+x)^{\frac{1}{3}}}$

$$= \frac{(a^2 - x^2)^{\frac{3}{2}}}{a^{\frac{3}{2}}(a+x)^{\frac{3}{3}}} = \frac{(a+x)^{\frac{3}{2}}(a-x)^{\frac{3}{2}}}{a^{\frac{3}{2}}(a+x)} = \frac{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}{a^{\frac{3}{2}}}.$$

Ex. 3. The fourth power of $\frac{ab}{x^2}\sqrt[5]{(c+x)^3}$ = the fourth power of $\frac{ab}{x^2}(c+x)^{\frac{3}{5}} = \frac{a^4b^4}{x^8}(c+x)^{\frac{12}{5}} = \frac{a^4b^4(c+x)^2}{x^8}\sqrt[5]{(c+x)^2}$.

Ex. 4. Required the square, cube, &c. of $a-b\sqrt{x}$.

Here the root $= a - bx^{\frac{1}{2}}$

$$\begin{array}{l} a - bx^{\frac{1}{2}} \\ \hline a^2 - abx^{\frac{1}{2}} \\ \quad - abx^{\frac{1}{2}} + b^2x \\ \hline \end{array}$$

$\therefore$ the square $= a^2 - 2abx^{\frac{1}{2}} + b^2x$;

$$\begin{array}{l} a - bx^{\frac{1}{2}} \\ \hline a^3 - 2a^2bx^{\frac{1}{2}} + ab^2x \\ \quad - a^2bx^{\frac{1}{2}} + 2ab^2x - b^3x^{\frac{3}{2}} \\ \hline \end{array}$$

$\therefore$ the cube $= a^3 - 3a^2bx^{\frac{1}{2}} + 3ab^2x - b^3x^{\frac{3}{2}}$; &c.

113. Cor. Since $\sqrt[n]{a}$ raised to the $m^{\text{th}}$ power $= a^{\frac{m}{n}}$, if we suppose $n=m$ we shall have the $m^{\text{th}}$ power of $\sqrt[m]{a}=a$; whence it is inferred that the $m^{\text{th}}$ power of a surd whose index is $\frac{1}{m}$ is obtained by removing the index or radical sign.

Ex. 1. The square of the irrational quantity

$\sqrt{a^2+bx}$ is $(a^2+bx)^{\frac{1}{2}} \times (a^2+bx)^{\frac{1}{2}} = (a^2+bx)^{\frac{2}{2}} = a^2+bx$.

Ex. 2. The cube of $a\sqrt[3]{b^3-x^3+3x\sqrt[4]{ax^2}}$ is

$$a^3(b^3-x^3+3x\sqrt[4]{ax^2}) = a^3b^3 - a^3x^3 + 3a^3x\sqrt[4]{ax^2}.$$

## VII. EVOLUTION.

### 114. *To find the Roots of a Surd.*

Let $\sqrt[n]{a}$ be any proposed surd; then since it is clear that $\sqrt[n]{a}$ or $a^{\frac{1}{n}} = a^{\frac{1}{mn}+\frac{1}{mn}+\text{\&c. to } m \text{ terms}} = a^{\frac{1}{mn}} \times a^{\frac{1}{mn}} \times a^{\frac{1}{mn}} \times$ &c. to $m$ factors, it follows that the $m^{\text{th}}$ root of the proposed surd will be any one of these factors, or $a^{\frac{1}{mn}}$.

Hence any root of a surd is obtained by dividing its index by the number belonging to the radical sign indicating the required root, as in rational quantities.

Of rational quantities combined with surds the roots must be found separately and connected by the proper sign.

Ex. 1. The square root of $\sqrt{ax} = (ax)^{\frac{1}{2}\div 2} = (ax)^{\frac{1}{4}}$; the cube root $= (ax)^{\frac{1}{2}\div 3} = (ax)^{\frac{1}{6}}$; &c.

Ex. 2. The cube root of $\dfrac{\sqrt{a+x}}{\sqrt[5]{b^3}}$ which is the same as the cube root of

$$\frac{(a+x)^{\frac{1}{2}}}{b^{\frac{3}{5}}} = \frac{(a+x)^{\frac{1}{2}\div 3}}{b^{\frac{3}{5}\div 3}} = \frac{(a+x)^{\frac{1}{6}}}{b^{\frac{1}{5}}} = \frac{\sqrt[6]{a+x}}{\sqrt[5]{b}}.$$

Ex. 3. The fourth root of $\dfrac{a^4}{x^2}\sqrt[3]{x^2-ax} = \dfrac{a}{x^{\frac{1}{2}}}(x^2-ax)^{\frac{1}{12}}$

$$= \frac{ax^{\frac{1}{12}}(x-a)^{\frac{1}{12}}}{x^{\frac{1}{2}}} = \frac{a\,(x-a)^{\frac{1}{12}}}{x^{\frac{5}{12}}} = a\sqrt[12]{\frac{x-a}{x^5}}.$$

Ex. 4. Let it be required to find the square root of

$$a - 2\sqrt{a}\,\sqrt[3]{bx} + b^{\frac{2}{3}}x^{\frac{2}{3}}.$$

Here, proceeding as in integral quantities, we have the following operation:

$$a - 2\sqrt{a}\sqrt[3]{bx} + b^{\frac{2}{3}}x^{\frac{2}{3}}\,(\sqrt{a} - \sqrt[3]{bx},$$

$$\underline{a \qquad\qquad\qquad\qquad}$$

$$2\sqrt{a} - \sqrt[3]{bx})\; - 2\sqrt{a}\sqrt[3]{bx} + b^{\frac{2}{3}}x^{\frac{2}{3}}$$

$$\underline{- 2\sqrt{a}\sqrt[3]{bx} + b^{\frac{2}{3}}x^{\frac{2}{3}}}$$

so that the square root is $\sqrt{a} - \sqrt[3]{bx}$.

115. Cor. By a repetition of the process explained in the last article, the roots of the roots of surds may be similarly expressed: thus,

the $m^{\text{th}}$ root of the $n^{\text{th}}$ root of $\sqrt[p]{a}$ = the $m^{\text{th}}$ root of $a^{\frac{1}{np}} = a^{\frac{1}{mnp}}$,

that is, $\sqrt[m]{\sqrt[n]{\sqrt[p]{a}}}$ is equivalent to $a^{\frac{1}{mnp}}$:

so likewise $\sqrt[m]{\sqrt[n]{\sqrt[p]{\sqrt[q]{a}}}}$ is equivalent to $a^{\frac{1}{mnpq}}$; &c.

116. Though it may be impossible to extract the root of an algebraical surd as it stands, such a modification of it may frequently be made by the addition and subtraction of the same quantity, that the extraction may be effected by the ordinary methods.

Ex. 1. Extract the square root of $a^2 + 2x\sqrt{a^2 - x^2}$.

Here, adding and subtracting $x^2$, we have the following operation:

$$a^2 - x^2 + 2x\sqrt{a^2 - x^2} + x^2(\sqrt{a^2 - x^2} + x,$$
$$a^2 - x^2$$

$$2\sqrt{a^2 - x^2} + x) + 2x\sqrt{a^2 - x^2} + x^2$$
$$+ 2x\sqrt{a^2 - x^2} + x^2$$

so that the square root is $\sqrt{a^2 - x^2} + x$.

Ex. 2. To extract the cube root of the expression

$$2a + 3\sqrt[3]{a^2 - x^2}\,\{\sqrt[3]{a + x} + \sqrt[3]{a - x}\},$$

we have merely to add and subtract $x$, so that it becomes

$$a + x + 3\sqrt[3]{a^2 - x^2}\,\{\sqrt[3]{a + x} + \sqrt[3]{a - x}\} + a - x,$$

which is manifestly the cube of $\sqrt[3]{a + x} + \sqrt[3]{a - x}$, the required root.

117. The subject of the last article being merely an artifice must necessarily be precarious, and it still remains to explain the theory of the extraction of the roots of surds in those cases wherein the extraction can be effected in a determinate form. No universal rule can be given, but the following articles will enable us to find the square and cube roots of binomial quadratic surds, whenever they can be exhibited as surds of the same description.

118. The product of two quadratic surds not having the same irrational part, is irrational.

For, if possible, let $\sqrt{x} \times \sqrt{y} = m$, then squaring both sides we have $xy = m^2$, and $\therefore y = \dfrac{m^2}{x} = \dfrac{m^2 x}{x^2}$; whence it follows that $\sqrt{y} = \dfrac{m}{x}\sqrt{x}$, or the surds have the same irrational factor $\sqrt{x}$, which is contrary to the supposition.

119. The square root of a rational quantity can neither be equivalent to the sum or difference of a rational quantity and a surd, nor to the sum or difference of two or more surds not having the same irrational factor.

If $\sqrt{a}=x\pm\sqrt{y}$, then by squaring both sides, we have $a=x^2\pm 2x\sqrt{y}+y$; whence $\pm\sqrt{y}=\dfrac{a-x^2-y}{2x}$, or a surd is equivalent to a rational quantity, which is impossible.

Again, if $\sqrt{a}=\sqrt{x}\pm\sqrt{y}$, we shall have by the same process, $a=x\pm 2\sqrt{xy}+y$, and $\therefore\ \pm\sqrt{xy}=\dfrac{a-x-y}{2}$, which is also absurd.

120. Cor. 1. Hence, in the equation $a\pm\sqrt{b}=x\pm\sqrt{y}$, we must have $a=x$, and $\sqrt{b}=\sqrt{y}$; for if this were not the case, the square root of a rational quantity would be partly a rational quantity and partly a quadratic surd.

121. Cor. 2. It follows, therefore, that if $a+\sqrt{b}=x+\sqrt{y}$, the equation $a-\sqrt{b}=x-\sqrt{y}$, also holds good.

122. *To extract the square root of a binomial surd, one of whose terms is a rational quantity, and the other a quadratic surd.*

Let $a\pm\sqrt{b}$ be the proposed binomial surd, and assume $\sqrt{a\pm\sqrt{b}}=\sqrt{u}\pm\sqrt{v}$, the algebraical signs of the latter quantities being the same on both sides:

$$\therefore\ a\pm\sqrt{b}=u\pm 2\sqrt{uv}+v=(u+v)\pm 2\sqrt{uv}:$$

hence by (120) we have $u+v=a$, and $2\sqrt{uv}=\sqrt{b}$:

squaring both sides of these two equations, we obtain

$$u^2+2uv+v^2=a^2$$
$$\text{and } 4uv \qquad =b,$$

$\therefore$ by subtraction and evolution, we find

$$u^2 - 2uv + v^2 = a^2 - b, \text{ and } u - v = \sqrt{a^2 - b};$$

whence, by addition, subtraction, &c. we get

$$2u = a + \sqrt{a^2 - b}, \; 2v = a - \sqrt{a^2 - b},$$

$$\sqrt{u} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}}, \text{ and } \sqrt{v} = \sqrt{\frac{a - \sqrt{a^2 - b}}{2}},$$

$$\text{and } \therefore \sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}:$$

wherefore, in order that the square root of the proposed surd may be a binomial surd, one or both of whose terms are quadratic surds, the quantity $a^2 - b$ must be a complete square, and the required root is found.

Ex. 1. Extract the square root of $x - 2\sqrt{x-1}$.

By the substitution of $x$ and $2\sqrt{x-1}$ in the places of $a$ and $\sqrt{b}$ in the general formula above deduced, the square root required will be found to be $\sqrt{x-1} - 1$: but to avoid substitutions it is generally most convenient to perform the operation at length; thus,

$$\text{let } \sqrt{x - 2\sqrt{x-1}} = \sqrt{u} - \sqrt{v},$$

$$\text{so that } x - 2\sqrt{x-1} = u + v - 2\sqrt{uv}:$$

$$\therefore u + v = x, \text{ and } 2\sqrt{uv} = 2\sqrt{x-1}:$$

$$\text{whence we get } u^2 + 2uv + v^2 = x^2$$

$$\text{and } 4uv \quad = 4x - 4,$$

$$\therefore u^2 - 2uv + v^2 = x^2 - 4x + 4, \text{ and } u - v = x - 2:$$

$$\therefore 2u = 2x - 2, \; 2v = 2; \; \sqrt{u} = \sqrt{x-1}, \; \sqrt{v} = 1;$$

and the required square root $= \sqrt{x-1} - 1$, as before.

Ex. 2. Extract the square root of the expression

$$2+2(1-x)\sqrt{1+2x-x^2}.$$

Assume $\sqrt{2+2(1-x)\sqrt{1+2x-x^2}}=\sqrt{u}+\sqrt{v}$,

$$\therefore\ u+v+2\sqrt{uv}=2+2(1-x)\sqrt{1+2x-x^2};$$

whence $u+v=2$, and $2\sqrt{uv}=2(1-x)\sqrt{1+2x-x^2}$.

$$\therefore\ u^2+2uv+v^2=4$$

$$\text{and } 4uv \quad =4(1-4x^2+4x^3-x^4);$$

$\therefore\ u^2-2uv+v^2=16x^2-16x^3+4x^4$, and $u-v=4x-2x^2$:

$\therefore$ we have $u=1+2x-x^2$, and $\sqrt{u}=\pm\sqrt{1+2x-x^2}$,

also $v=1-2x+x^2$, and $\sqrt{v}=\pm(1-x)$:

whence the required root

$$=\sqrt{1+2x-x^2}+1-x, \text{ or } x-1-\sqrt{1+2x-x^2}.$$

Ex. 3. To extract the square root of the numerical surd $28+5\sqrt{12}$, we proceed exactly as before, and assume

$$\sqrt{28+5\sqrt{12}}=\sqrt{u}+\sqrt{v};$$

whence $u+v=28$, $2\sqrt{uv}=5\sqrt{12}$; $u^2+2uv+v^2=784$, and $4uv=300$, so that $u^2-2uv+v^2=484$, and $\therefore\ u-v=22$; $\therefore$ as before $u=25$, and $v=3$, and the required root $=5+\sqrt{3}$.

Ex. 4. Extract the square root of $\sqrt{32}-\sqrt{24}$.

Here, both the terms being surds, it is obvious that the general method cannot be applied to this quantity as it stands, and it will therefore be necessary to make it assume the proper form: thus, we have

$$\sqrt{32}-\sqrt{24}=\sqrt{8.4}-\sqrt{8.3}=\sqrt{8}(2-\sqrt{3});$$

and proceeding as before with the latter factor, we find its square root to be $\frac{\sqrt{3}-1}{\sqrt{2}}$, and thence the square root required will manifestly be

$$\sqrt{\sqrt{8}}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)=\sqrt[4]{8}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)=\sqrt[4]{\frac{8}{4}}(\sqrt{3}-1)$$

$$=\sqrt[4]{2}(\sqrt{3}-1)=\sqrt[4]{18}-\sqrt[4]{2}.$$

123. Cor. An assumption similar to what has been adopted in the last article, will frequently enable us to extract the square root of a surd involving more than two terms: thus,

let $\sqrt{6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}}=\sqrt{u}+\sqrt{v}+\sqrt{w}$,

$\therefore$ we get

$$6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}=u+v+w+2\sqrt{uv}+2\sqrt{uw}+2\sqrt{vw},$$

whence we have $u+v+w=6$,

$$2\sqrt{uv}=2\sqrt{2},\quad 2\sqrt{uw}=2\sqrt{3},\quad 2\sqrt{vw}=2\sqrt{6};$$

and squaring both sides of each of these latter equations, we obtain

$$uv=2,\quad uw=3,\quad \text{and}\quad vw=6;$$

whence, by multiplication, $u^2v^2w^2=36$, and $\therefore$ $uvw=6$;

$$\therefore\ u=\frac{uvw}{vw}=\frac{6}{6}=1,\quad v=\frac{uvw}{uw}=\frac{6}{3}=2,\quad w=\frac{uvw}{uv}=\frac{6}{2}=3,$$

and since the sum of $u$, $v$ and $w$ is 6, the required square root is $1+\sqrt{2}+\sqrt{3}$.

124. It may be proved as in (119), that the cube root of a rational quantity cannot be equivalent to the sum or difference of a rational quantity and a surd, nor to the sum or difference of two or more surds.

Also, if $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, we shall, by involution, have $a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$; whence, by (120),

$$a=x^3+3xy, \quad \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y},$$

and $\therefore\ a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}=(x-\sqrt{y})^3$;

wherefore we have $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$: that is,

if $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, then will $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$.

125. *To extract the cube root of a binomial surd, one of whose terms is a rational quantity, and the other a quadratic surd.*

Let $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, so that $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$, $\therefore$ by multiplying together the corresponding members of these equations, we have

$$\sqrt[3]{a^2-b}=x^2-y, \text{ or } a^2-b=(x^2-y)^3:$$

let $a^2-b=c^3$, a perfect cube, then $x^2-y=c$, or $y=x^2-c$;

but since from the last article $a=x^3+3xy$, we have

$$a=x^3+3x(x^2-c)=4x^3-3cx;$$

and because $x$ is supposed to be rational, its value may easily be found by trial, and therefore $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{x^2-c}$ will be expressed in terms of $a$ and $b$.

If $a^2-b$ be not a complete cube, the root cannot be exhibited in the same form as the surd itself, and recourse must be had to expedients hereafter explained.

Ex. Extract the cube root of the numerical surd

$$20 \pm 14\sqrt{2}.$$

Assume $\sqrt[3]{20 \pm 14\sqrt{2}}=x \pm \sqrt{y}$, whence we have

$$x^2 - y = \sqrt[3]{(20 + 14\sqrt{2})(20 - 14\sqrt{2})}$$

$$= \sqrt[3]{400 - 392} = \sqrt[3]{8} = 2;$$

$\therefore\ y = x^2 - 2$, and $20 = 4x^3 - 6x$, from which $x$ is immediately found by trial to be 2, and $\therefore\ y = 4 - 2 = 2$;

$$\text{hence } \sqrt[3]{20 \pm 14\sqrt{2}} = 2 \pm \sqrt{2}, \text{ the required root.}$$

126. The fourth root of a binomial surd may be determined by extracting the square root of its square root; and by a continuation of the same process, the $(2^m)^{\text{th}}$ root may be found when it is possible: similarly the sixth root, being the square root of the cube root, may sometimes be found by the methods above explained, but for a more general theory of the subject the reader is referred to the latter part of Chap. VII.

127. Though the value of a surd can never be accurately exhibited in terms of rational quantities, approximations may be made to its true value to any degree of exactness required in practice.

Let $\sqrt{N}$ be the proposed surd, and suppose $a^2$ to be the greatest square number contained in $N$, and $b$ a quantity such that

$$N = a^2 + 2ab + b^2:$$

then since $b$ is necessarily a proper fraction, $b^2$ is small when compared with $a^2 + 2ab$, so that $N = a^2 + 2ab$ nearly;

$$\therefore\ b = \frac{N - a^2}{2a} \text{ nearly,}$$

$$\text{whence } \sqrt{N} = a + b = a + \frac{N - a^2}{2a} = \frac{a^2 + N}{2a} \text{ nearly,}$$

which is evidently somewhat greater than the true value:

Call this value $a'$, and let $b'$ be so assumed that

$$N=(a'-b')^2=a'^2-2a'b'+b'^2,$$

$$\text{then as before } b'=\frac{a'^2-N}{2a'} \text{ nearly;}$$

$$\therefore \sqrt{N}=a'-b'=a'-\frac{a'^2-N}{2a'}=\frac{a'^2+N}{2a'} \text{ nearly,}$$

which is obviously nearer the true value: and by a continuation of this process, we shall approximate more and more nearly to the true value of the surd, till the required degree of exactness be attained.

A similar result might have been obtained by assuming $a^2$ to represent the square number next greater than $N$.

Ex. To find the approximate values of $\sqrt{2}$, we have $N=2$, $a=1$,

$$\text{whence } \sqrt{2}=\frac{1+2}{2}=\frac{3}{2} \text{ nearly,}$$

which is the first approximation and is greater than the true root:

$$\text{also, } \because \ a'=\frac{3}{2}, \text{ we have}$$

$$\sqrt{2}=\frac{\frac{9}{4}+2}{3}=\frac{17}{12} \text{ nearly,}$$

the second approximation, which is more nearly the true value; and the third approximation will be found to be $\frac{577}{408}$, which is nearer still; and so on.

128. Cor. Since by (30) the number of figures composing the square of any quantity cannot exceed twice the number in the quantity itself, it follows that if $a$ contain at least one figure more than $b$ does, and as many cyphers after it as there are figures in $b$, $b^2$ cannot contain as many figures

as there are in $2a$, and $\therefore \frac{b^2}{2a}$ is necessarily a proper fraction;

$$\text{whence } b = \frac{N - a^2}{2a}$$

may be obtained as far as the unit's place by division only.

Ex. Let it be required to extract the square root of 2.

Proceeding at first in the ordinary method, and then by division, we have

$$
\begin{array}{rl}
 & \dot{2}.0\dot{0}0\dot{0}0\dot{0}\ (1.414 \\
 & 1 \\
\hline
24) & 100 \\
 & \ \ 96 \\
\hline
281) & 400 \\
 & 281 \\
\hline
2824) & 11900 \\
 & 11296 \\
\hline
2828) & 6040000\ (2135 \\
 & 5656 \\
\hline
 & \ \ 3840 \\
 & \ \ 2828 \\
\hline
 & \ \ 10120 \\
 & \ \ \ \ 8484 \\
\hline
 & \ \ \ \ 16360 \\
 & \ \ \ \ 14140 \\
\hline
 & \ \ \ \ \ \ 2220, \\
\hline
\end{array}
$$

so that the approximate root is 1.4142135 &c. the number of figures, after four places in the root are ascertained, being doubled by division only.

Similar methods may with different degrees of accuracy be made use of in approximating to the values of cubic, biquadratic, &c. surds.

## VIII. TRANSFORMATION.

129. A surd may be converted into a series by a continuation of the operation indicated to any extent we please, and it is obvious from (23) that the entire result would consist of an infinite number of terms: it may also be further remarked, that it is seldom possible to discover the law according to which the successive terms are formed. One example of this will suffice.

Ex. To convert $\sqrt{a^2+x^2}$ into an infinite series.

$$a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-\&c.\right.$$

$$a^2$$

$$\left.2a+\frac{x^2}{2a}\right)\;+x^2$$

$$x^2+\frac{x^4}{4a^2}$$

$$\left.2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\right)\;-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$\left.2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{16a^5}\right)\;\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\&c.$$

$$-\frac{5x^8}{64a^6}+\&c.$$

whence we have $\sqrt{a^2+x^2} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \&c.$ *in infinitum*, where the sign = is used in the sense in which it has been explained in (88).

A monomial surd as $\sqrt{a} = \sqrt{1+(a-1)}$, a trinomial surd as $\sqrt{1+x+x^2} = \sqrt{1+x(1+x)}$, &c. may be expressed after a similar manner.

130. The operations of Multiplication and Division will enable us to exhibit any proposed surd in different forms, without altering its value.

Ex. 1. If we have the surd $\frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$, and multiply the numerator and denominator by $\sqrt{a+x}-\sqrt{a-x}$, there will result the equivalent surd

$$\frac{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})}{(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})} = \frac{2a-2\sqrt{a^2-x^2}}{2x},$$

which, by dividing the terms by 2, becomes $= \frac{a-\sqrt{a^2-x^2}}{x}$:

Again, multiplying the numerator and denominator by $\sqrt{a+x}+\sqrt{a-x}$, we find it equivalent to

$$\frac{(\sqrt{a+x}-\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}+\sqrt{a-x})}$$

$$= \frac{2x}{2a+2\sqrt{a^2-x^2}} = \frac{x}{a+\sqrt{a^2-x^2}}.$$

Ex. 2. To reduce the fractional surd $\frac{1+x^{\frac{1}{2}}-x^{\frac{3}{2}}-x^2}{x^{\frac{3}{2}}+x^2+5x^2+5x^{\frac{5}{2}}}$ to its lowest terms, we have merely to divide its numerator and

denominator by their greatest common measure $1+x^{\frac{1}{2}}$ determined as in rational quantities: whence the equivalent surd in the least terms will be $\dfrac{1-x^{\frac{3}{2}}}{x^{\frac{3}{2}}+5x^2}$.

Ex. 3. To reduce $\dfrac{1}{\sqrt{1+x}}$, $\dfrac{2x}{(1+x)^{\frac{3}{2}}}$ and $\dfrac{3x^2}{(1+x)^{\frac{5}{2}}}$ to equivalent surds having a common denominator, we observe that $(1+x)^{\frac{5}{2}}$ is the least common multiple of the proposed denominators found as in rational quantities: whence as in (72) the new numerators will be $(1+x)^2$, $2x+2x^2$ and $3x^2$ respectively: and the corresponding new fractions

$$\frac{(1+x)^2}{(1+x)^{\frac{5}{2}}}, \quad \frac{2x+2x^2}{(1+x)^{\frac{5}{2}}} \text{ and } \frac{3x^2}{(1+x)^{\frac{5}{2}}}.$$

In the same manner surds of this description are prepared for the operations of Addition and Subtraction.

131. By effecting the operation of Involution, and indicating the reverse one of Evolution, surds and mixed quantities may be transformed into other surds of a more general form.

Ex. 1. By the operation of involution, we have

$$(\sqrt{x}+\sqrt{1-x})^2 = 1+2\sqrt{x-x^2},$$

whence it obviously follows that $\sqrt{x}+\sqrt{1-x}$ is equivalent to $\sqrt{1+2\sqrt{x-x^2}}$.

Ex. 2. Let $x-y+\sqrt{2xy-y^2}$ be a proposed mixed quantity, then we shall have immediately

$$(x-y+\sqrt{2xy-y^2})^2 = x^2-2xy+y^2+2(x-y)\sqrt{2xy-y^2}$$
$$+2xy-y^2 = x^2+2(x-y)\sqrt{2xy-y^2};$$

$$\therefore\ x-y+\sqrt{2xy-y^2}=\sqrt{x^2+2(x-y)\sqrt{2xy-y^2}},$$

in which all the terms are under the radical sign: and the original quantity, which is partly rational and partly surd, is thus expressed as what is called a general surd.

Ex. 3. Again, by the operations of involution and evolution,

$$\sqrt[4]{a+\sqrt{b}}+\sqrt[4]{a-\sqrt{b}}=\sqrt{\{\sqrt[4]{a+\sqrt{b}}+\sqrt[4]{a-\sqrt{b}}\}^2}$$

$$=\sqrt{\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}+2\sqrt[4]{a^2-b}}:$$

$$\text{but } \sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}$$

$$=\sqrt{\{\sqrt{a+\sqrt{b}}+\sqrt{a-\sqrt{b}}\}^2}=\sqrt{2a+2\sqrt{a^2-b}};$$

$\therefore$ by the substitution of this equivalent quantity in the preceding expression, we shall have

$$\sqrt[4]{a+\sqrt{b}}+\sqrt[4]{a-\sqrt{b}}=\sqrt{\sqrt{2a+2\sqrt{a^2-b}}+2\sqrt[4]{a^2-b}},$$

which, by means of fractional indices, may be written in the form

$$\{(2a+2(a^2-b)^{\frac{1}{2}})^{\frac{1}{2}}+2(a^2-b)^{\frac{1}{4}}\}^{\frac{1}{2}}.$$

A similar expression may be deduced for

$$\sqrt[2m]{a+\sqrt{b}}+\sqrt[2n]{a-\sqrt{b}}.$$

132. Since by the nature of the operation indicated, if

$$\sqrt[m]{a} \text{ or } a^{\frac{1}{m}}$$

be raised to the $m^{\text{th}}$ power, the result is the rational quantity $a$, it is manifest that by transposing all the rational quantities to the same side, according to the rule laid down in (44), an equation may be *cleared of Surds* by the proper involutions of both its members.

Ex. 1. To clear of surds the equation $ax = b + \sqrt{cx}$, we have $ax - b = \sqrt{cx}$, and squaring both sides, we get

$$a^2x^2 - 2abx + b^2 = cx, \text{ which is free from surds.}$$

Ex. 2. If we have $\sqrt{2ax + x^2 + \sqrt{a^2 + x^2}} = a - x$, then will

$$2ax + x^2 + \sqrt{a^2 + x^2} = a^2 - 2ax + x^2,$$

and by rejecting $x^2$ from both sides and transposing $2ax$, we have

$$\sqrt{a^2 + x^2} = a^2 - 4ax;$$

whence squaring again we obtain

$$a^2 + x^2 = a^4 - 8a^3x + 16a^2x^2,$$

which does not involve a surd.

133. From what has been already said, it is evident that any simple surd being continually multiplied into itself, will at length give a rational result, and that compound surds will, by a similar operation, give results still involving irrational quantities.

For every compound surd, however, there exists another compound surd, which being multiplied into it will give a rational product: thus, if the surd

$$\sqrt{a} \pm \sqrt{b}$$

be multiplied by the surd

$$\sqrt{a} \mp \sqrt{b},$$

there results the rational quantity

$$a - b;$$

and the following article contains the general investigation of the multipliers, which will rationalize any surds whatever.

134. It has been seen that by actual division

$$\frac{x^m - y^m}{x - y} = x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1},$$

whatever whole number $m$ may represent, the number of terms being also $m$; whence it obviously follows that

$$(x-y)(x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1}) = x^m - y^m:$$

now if $x$ and $y$ represent any two surds, it is manifest that the latter side will be a rational quantity, whenever $m$ is assumed of such a magnitude as to render both $x^m$ and $y^m$ rational, and the corresponding rationalizing multiplier will be

$$x^{m-1} + x^{m-2}y + \&c. + xy^{m-2} + y^{m-1}.$$

If the sign of the latter surd be positive, we shall have

$$(x+y)(x^{m-1} - x^{m-2}y + \&c. \mp xy^{m-2} \pm y^{m-1}) = x^m \pm y^m,$$

where the upper or lower sign is to be used according as $m$ is odd or even: wherefore in this case the rationalizing multiplier will be

$$x^{m-1} - x^{m-2}y + \&c. \mp xy^{m-2} \pm y^{m-1},$$

and the corresponding rational result $x^m \pm y^m$.

Ex. 1. Required the surd multiplier which will render $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ a rational quantity.

Here it is obvious that $m = 4$, and therefore the multiplier will be

$$a^{\frac{3}{4}\times 3} + a^{\frac{3}{4}\times 2}b^{\frac{3}{4}\times 1} + a^{\frac{3}{4}\times 1}b^{\frac{3}{4}\times 2} + b^{\frac{3}{4}\times 3},$$

$$\text{or } a^{\frac{9}{4}} + a^{\frac{3}{2}}b^{\frac{3}{4}} + a^{\frac{3}{4}}b^{\frac{3}{2}} + b^{\frac{9}{4}};$$

and the rationalized result is $a^3 - b^3$.

Ex. 2. What is the surd multiplier requisite to make $a^{\frac{1}{2}} + b^{\frac{1}{3}}$ a rational quantity?

In this case $m$ evidently $= 6$, and therefore

$$a^{\frac{1}{2}\times 5} - a^{\frac{1}{2}\times 4} b^{\frac{1}{3}\times 1} + a^{\frac{1}{2}\times 3} b^{\frac{1}{3}\times 2} - a^{\frac{1}{2}\times 2} b^{\frac{1}{3}\times 3} + a^{\frac{1}{2}\times 1} b^{\frac{1}{3}\times 4} - b^{\frac{1}{3}\times 5},$$

$$\text{or } a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - ab + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}}$$

is the multiplier sought, and the resulting rational quantity will be $a^3 - b^2$.

135. By means of this theorem a multiplier may be found, which will render rational any binomial surd whatever, and by a continuation of a similar process the same effects may be produced upon surds consisting of three or more terms.

Ex. Let $\sqrt{a} + \sqrt{b} + \sqrt{c}$ be the surd proposed, then we have

$$\{(\sqrt{a} + \sqrt{b}) + \sqrt{c}\}\{(\sqrt{a} + \sqrt{b}) - \sqrt{c}\}$$

$$= (\sqrt{a} + \sqrt{b})^2 - (\sqrt{c})^2 = (a + b - c) + 2\sqrt{ab}:$$

and again $\{(a+b-c) + 2\sqrt{ab}\}\{(a+b-c) - 2\sqrt{ab}\}$

$$= (a+b-c)^2 - 4ab = a^2 + 2ab + b^2 - 2(a+b)c + c^2 - 4ab$$

$= a^2 + b^2 + c^2 - 2(ab + ac + bc)$, which is rational.

136. The principles explained in the preceding articles will manifestly enable us to clear fractional surds of irrational denominators.

Thus, in the fraction $\dfrac{\sqrt{x^2+x+1} + \sqrt{x^2-x-1}}{\sqrt{x^2+x+1} - \sqrt{x^2-x-1}}$, if we multiply both the numerator and denominator by the numerator, the result will be

$$\frac{x^2 + \sqrt{x^4 - x^2 - 2x - 1}}{x + 1}.$$

The division of one surd by another may also be simplified by the same method.

Thus, the quotient of $a+\sqrt{b}$ by $c-\sqrt{d}=\dfrac{a+\sqrt{b}}{c-\sqrt{d}}$

$$=\frac{(a+\sqrt{b})(c+\sqrt{d})}{(c-\sqrt{d})(c+\sqrt{d})}=\frac{ac+a\sqrt{d}+c\sqrt{b}+\sqrt{bd}}{c^2-d}:$$

similarly, the quotient of $5-7\sqrt{3}$ by $1+\sqrt{3}=\dfrac{5-7\sqrt{3}}{1+\sqrt{3}}$

$$=\frac{(5-7\sqrt{3})(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}=\frac{26-12\sqrt{3}}{-2}=6\sqrt{3}-13.$$

## IX. IMAGINARY OR IMPOSSIBLE QUANTITIES.

137. From (22) it appears that every quantity whether positive or negative when raised to an even power, gives a positive result, and thence it follows that no even root of a negative quantity can be either positive or negative: the even roots of negative quantities can therefore be only *indicated* or *expressed*, and are usually termed *Imaginary* or *Impossible*.

Thus, the square root of $-a^2$ being neither $+a$ nor $-a$, is written $\sqrt{-a^2}$, and is equivalent to

$$\sqrt{a^2\times(-1)}=\sqrt{a^2}\sqrt{-1}=\pm a\sqrt{-1},$$

which is said to involve the imaginary or impossible quantity $\sqrt{-1}$.

Similarly, we shall manifestly have

$$\begin{aligned}\sqrt[4]{-a^4}&=\sqrt[4]{a^4\times(-1)}=\sqrt[4]{a^4}\sqrt[4]{-1}\\&=\pm a\sqrt[4]{-1}=\pm a\sqrt{\sqrt{-1}};\\\sqrt[6]{-a^6}&=\sqrt[6]{a^6\times(-1)}=\sqrt[6]{a^6}\sqrt[6]{-1}\\&=\pm a\sqrt[6]{-1}=\pm a\sqrt[3]{\sqrt{-1}};\\\&c.\quad&=\quad\&c.\end{aligned}$$

$$\sqrt[2m]{-a^{2m}} = \sqrt[2m]{a^{2m} \times (-1)} = \sqrt[2m]{a^{2m}}\sqrt[2m]{-1}$$
$$= \pm a\sqrt[2m]{-1} = \pm a\sqrt[m]{\sqrt{-1}}:$$

and it is hence obvious that every quantity of the kind above described is reducible to the determination of the various roots of the expression $\sqrt{-1}$; or in other words, that the reality or impossibility of all such expressions depends upon the reality or impossibility of $\sqrt{-1}$ and its roots.

138. By actual evolution, it is easily proved as in (129) that

$$\sqrt{x^2-1} = x - \frac{1}{2x} - \frac{1}{8x^3} - \frac{1}{16x^5} - \&c. \textit{ in infinitum}:$$

whence if $x$ be supposed $=0$, we shall manifestly have

$$\sqrt{-1} = 0 - \frac{1}{0} - \frac{1}{0} - \frac{1}{0} - \&c. \textit{ in infinitum},$$
$$= 0 - \infty - \infty - \infty - \&c. \textit{ in infinitum} \text{ by (89)}:$$

and since to this expression no definite meaning can be attached, the value of $\sqrt{-1}$ cannot be assigned *arithmetically* either accurately or approximately, and on this account may be termed *unassignable* with greater propriety than either *imaginary* or *impossible*.

139. All the Arithmetical operations upon what are termed imaginary quantities will therefore depend upon the treatment of the symbol $\sqrt{-1}$, which, due regard being paid to the manner in which it originated, will obviously be the same as that required for any irrational quantity. At all events, since every imaginary quantity is reducible so as to involve the same imaginary factor $\sqrt{-1}$, the only peculiarity that can occur, must be in their Multiplication and Involution, and this will very easily be removed by the consideration that the symbol $\sqrt{-1}$ is merely the *indication* of an operation which cannot be *effected* in assignable terms: thus,

$$\sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2,$$

and it is obvious that the operation here indicated by the radical sign is neutralized by the reverse operation whose index is 2, so that we have

$$\sqrt{-1} \times \sqrt{-1} = -1,$$

and not $\sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = \pm 1,$

as would have been the case, had the operation indicated been supposed capable of being effected.

Again,

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = (-1) \times \sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1.$$

Similarly, we shall have

$$(\sqrt{-1})^{4m} = \{(\sqrt{-1})^4\}^m = 1^m = 1:$$

$$(\sqrt{-1})^{4m+1} = (\sqrt{-1})^{4m} \times \sqrt{-1} = 1 \times \sqrt{-1} = \sqrt{-1}:$$

$$(\sqrt{-1})^{4m+2} = (\sqrt{-1})^{4m} \times (\sqrt{-1})^2 = 1 \times (-1) = -1:$$

$$(\sqrt{-1})^{4m+3} = (\sqrt{-1})^{4m} \times (\sqrt{-1})^3 = 1 \times (-\sqrt{-1}) = -\sqrt{-1}.$$

140. Retaining the same views, we shall readily obtain

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{a}\sqrt{-1} \times \sqrt{b}\sqrt{-1}$$

$$= \sqrt{ab}(\sqrt{-1})^2 = -\sqrt{ab}, \text{ which is possible:}$$

$$\sqrt[4]{-a} \times \sqrt[4]{-b} = \sqrt[4]{a}\sqrt[4]{-1} \times \sqrt[4]{b}\sqrt[4]{-1}$$

$$= \sqrt[4]{ab}(\sqrt[4]{-1})^2 = \sqrt[4]{ab}\sqrt{-1}, \text{ which is impossible:}$$

$$\sqrt[6]{-a} \times \sqrt[6]{-b} = \sqrt[6]{a}\sqrt[6]{-1} \times \sqrt[6]{b}\sqrt[6]{-1}$$

$$= \sqrt[6]{ab}(\sqrt[6]{-1})^2 = \sqrt[6]{ab}\sqrt[3]{-1} = -\sqrt[6]{ab}, \text{ which is possible:}$$

&c.................. = &c.........................................

$$\sqrt[2m]{-a} \times \sqrt[2m]{-b} = \sqrt[2m]{a}\sqrt[2m]{-1} \times \sqrt[2m]{b}\sqrt[2m]{-1}$$

$$= \sqrt[2m]{ab}(\sqrt[2m]{-1})^2 = \sqrt[2m]{ab}\sqrt[m]{-1},$$

which will manifestly be possible or impossible according as $m$ is an odd or even number.

141. All the other operations are performed exactly as in surds; and the observations made in articles (98), &c. are all rendered applicable to the description of quantities just considered by using the words *imaginary* or *impossible* in the place of *irrational* or *surd*.

It is, moreover, clear that every result, arrived at through the intervention of the symbol $\sqrt{-1}$ thus employed, must be correct, because no meaning is attached to it in any one step of an operation which it does not retain throughout: and we may observe that the doctrine of imaginary quantities to which it gives rise is of the utmost importance in Algebra, inasmuch as it frequently enables us to judge with certainty of the possibility or impossibility of a question proposed, and to establish in the higher parts of the Science, relations that could not otherwise be easily discovered.

142. We shall draw the present Chapter to a conclusion by proving that in the Addition, Subtraction, Multiplication, Division, Involution, and Evolution when possible, of quantities of the form $a \pm b\sqrt{-1}$, the results are always of the form $A \pm B\sqrt{-1}$.

(1). In Addition:

$$(a \pm b\sqrt{-1}) + (c \pm d\sqrt{-1}) + \&c.$$

$$= (a + c + \&c.) \pm (b + d + \&c.)\sqrt{-1}.$$

(2). In Subtraction:

$$(a \pm b\sqrt{-1}) - (c \pm d\sqrt{-1}) = (a - c) \pm (b - d)\sqrt{-1}.$$

(3). In Multiplication:

$$(a \pm b\sqrt{-1}) \times (c \pm d\sqrt{-1}) = (ac - bd) \pm (ad + bc)\sqrt{-1}.$$

(4). In Division:

$$(a \pm b\sqrt{-1}) \div (c \pm d\sqrt{-1}) = \frac{a \pm b\sqrt{-1}}{c \pm d\sqrt{-1}}$$

$$= \frac{(a \pm b\sqrt{-1})(c \mp d\sqrt{-1})}{(c \pm d\sqrt{-1})(c \mp d\sqrt{-1})} = \frac{ac + bd \pm (bc - ad)\sqrt{-1}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} \pm \frac{bc - ad}{c^2 + d^2}\sqrt{-1}.$$

(5). In Involution:

$$(a \pm b\sqrt{-1})^2 = a^2 \pm 2ab\sqrt{-1} - b^2 = (a^2 - b^2) \pm 2ab\sqrt{-1};$$

$$(a \pm b\sqrt{-1})^3 = a^3 \pm 3a^2b\sqrt{-1} - 3ab^2 \mp b^3\sqrt{-1}$$

$$= (a^3 - 3ab^2) \pm (3a^2b - b^3)\sqrt{-1};$$

&c............ = &c..................................

(6). In Evolution:

$$\sqrt{a \pm b\sqrt{-1}} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 + b^2}}{2}}$$

$$= \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} \pm \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}\sqrt{-1},$$

by (122), which, if $a^2 + b^2$ be a complete square, will manifestly be exhibited in the proposed form.

Similarly of the cube, &c. roots, when they can be determined.

# CHAP. VI.

*On the Solution of Equations involving one unknown quantity. On Elimination, and the Solution of Equations involving two or more unknown quantities.*

143. It has been observed in article (2), that the quantities employed in the science of Algebra are distinguished into known or given, and unknown or required, known quantities being generally denoted by the former letters of the alphabet $a$, $b$, $c$, &c. and unknown by the latter $x$, $y$, $z$, &c.; this distinction is chiefly confined to *Equations* of which we come now to treat, and the *Solution* of equations is the expressing the values of the latter in terms of, or by means of, the former.

The solutions of equations will be verified, if on substituting their values for $x$, $y$, $z$, &c. both sides become *identical:* and these quantities, which are termed the *Roots* of the equations, or values of the unknown quantities, are said thus to fulfil or satisfy the conditions which they involve.

Ex. 1. In the equation $4x+2=3x+4$, the letter $x$ denotes the unknown quantity which is combined with the given numbers 2, 3, and 4; and the solution of this equation will be effected, if we can find such a numerical value of $x$ as will render identical its two members $4x+2$ and $3x+4$.

A little consideration will, in this instance, shew that the value of $x$ must be 2, as this manifestly gives

$$4\times 2+2=3\times 2+4, \quad \text{or} \quad 10=10;$$

that is, the number 2 renders the two members of the equation identical, and therefore satisfies the condition expressed by it: and it will appear upon trial that no other number can.

Ex. 2. If there be proposed the equation

$$x^2 + 2 = 3x,$$

where the quantity $x$ and its square are combined with the known numerical magnitudes 1, 2, 3, it will be possible, but not without greater difficulty, to assign such values to $x$ as will satisfy the condition which it involves: thus,

if $x = 1$, we have $1^2 + 2 = 3 \times 1$, or $3 = 3$,

which is an identity: again,

if $x = 2$, we get $2^2 + 2 = 3 \times 2$, or $6 = 6$,

which is also an identity; that is, the numbers 1 and 2 are roots of this equation: but, besides these two numbers 1 and 2, no other quantity can be found possessing the same property of fulfilling the condition expressed by the equation

$$x^2 + 2 = 3x.$$

144. Similar trials might be made in other cases, but it is obvious that when the terms of the equation are numerous, and the unknown quantity and its powers and roots are much involved with those that are known, the mode above adopted being regulated by no rule, would be entirely incompetent to effect the solution.

As a first step, therefore, towards the solution of equations, we shall premise the two following self-evident propositions, namely,

(1). If equal quantities be added to, or subtracted from, equal quantities, the sums or differences are equal:

(2). If equal quantities be multiplied by, or divided by, equal quantities, the products or quotients are equal:

and by means of them shew how such an arrangement of the terms of an equation may be made, that the unknown quantity and its powers or roots may occupy exclusively one of the members, while the other is made up of such quantities as are supposed already known.

145. By means of Axiom (1) just recited, it has been proved in (44) that any quantity may be *transposed* from either side of an equation to the other, merely by changing its algebraical sign from $+$ to $-$, or from $-$ to $+$.

Ex. 1. Let the proposed equation be

$$4x-2=3x+4;$$

then it is obvious that $4x-3x=4+2$, or which is the same thing, $x=6$, as appears from effecting the operations indicated.

Ex. 2. If the equation proposed be

$$ax-b=cx+d,$$

$$\text{then will}\quad ax-cx=d+b=b+d,$$

which may obviously be written

$$(a-c)\,x=b+d;$$

and this may evidently stand also, if necessary, in the following form,

$$(a-c)\,x-(b+d)=0.$$

Ex. 3. Given $ax^2+bx-c=dx^2-ex-f$; then we shall have immediately

$$ax^2+bx-dx^2+ex=c-f,$$

$$\text{or}\quad (a-d)\,x^2+(b+e)\,x=c-f;$$

which may be written also in the form

$$(a-d)\,x^2+(b+e)\,x-(c-f)=0.$$

Similar steps may be taken, whatever powers of the unknown quantity are involved; and it is usual to arrange the terms according to its descending powers, the absolute or known term occupying exclusively the second side of the equation, or the last place in the first.

146. If fractional quantities be involved in the terms of an equation, it has been seen in (96) how, by means of Axiom 2, the equation may be cleared of fractions by multiplying all the terms by the least common multiple of their denominators.

Ex. 1. In the equation

$$\frac{x}{3} + 6 = \frac{2x}{5} + 3,$$

if we multiply all the terms by 15, the least common multiple of 3 and 5, as determined by (62), we shall obtain

$$5x + 90 = 6x + 45,$$

which does not involve fractional coefficients of the unknown quantity.

Ex. 2. If $\dfrac{a}{b-x} = \dfrac{b}{a+x}$, we shall obviously have

$$a(a+x) = b(b-x), \text{ or } a^2 + ax = b^2 - bx,$$

by clearing the equation of fractions; whence by the last article is obtained

$$ax + bx = b^2 - a^2, \text{ or } (a+b)x = b^2 - a^2,$$

$$\text{or } (a+b)x + a^2 - b^2 = 0.$$

Ex. 3. Let $\dfrac{2x}{3} - \dfrac{1-\frac{1}{2}x}{4x} = \dfrac{x-1}{2} + \dfrac{5x}{6} + \dfrac{7}{12}$, which since $\dfrac{1-\frac{1}{2}x}{4x} = \dfrac{2-x}{8x}$, will manifestly be equivalent to

$$\frac{2x}{3} - \frac{2-x}{8x} = \frac{x-1}{2} + \frac{5x}{6} + \frac{7}{12};$$

then, if both members be multiplied by $24x$ the least common multiple of the denominators, we obtain

$$16x^2 - 6 + 3x = 12x^2 - 12x + 20x^2 + 14x;$$

$$\therefore\ 16x^2 - 12x^2 - 20x^2 + 3x + 12x - 14x = 6,$$

$$\text{or}\quad -16x^2 + x = 6:$$

$$\text{whence}\quad -6 = 16x^2 - x,\ \text{or}\ 16x^2 - x = -6,$$

$$\text{or}\quad 16x^2 - x + 6 = 0,$$

which involves the same condition as the proposed equation, in a form divested of fractional coefficients of the unknown quantity.

147. If the unknown quantity in an equation appear in an irrational form, we have seen in (132) how, by an extended application of Axiom 2, the equation may be cleared of surds by proper transpositions and involutions of both its members.

Ex. 1. Let $\sqrt{x+3} = \sqrt{3x}$; then by squaring both members, we get

$$x + 3 = 3x,$$

which is free from surds, and may manifestly be written in any of the forms

$$3 = 2x,\quad 3 - 2x = 0,\quad \text{or}\quad 2x - 3 = 0.$$

Ex. 2. Given the equation $2a + \sqrt{2ax + x^2} = 2x$;

then by (145) $\sqrt{2ax + x^2} = 2x - 2a$;

whence, by equal involution of both members, is obtained

$$2ax + x^2 = 4x^2 - 8ax + 4a^2;$$

$$\therefore\ x^2 - 4x^2 + 2ax + 8ax = 4a^2,$$

$$\text{or}\quad -3x^2 + 10ax = 4a^2;$$

whence, changing the signs of both sides, which is equivalent to multiplying them by $-1$, we have

$$3x^2 - 10ax = -4a^2,\ \text{or}\ 3x^2 - 10ax + 4a^2 = 0.$$

Ex. 3. Let the equation be $\frac{\sqrt[3]{a+x}}{a} + \frac{\sqrt[3]{a+x}}{x} = \left(\frac{x}{b^2}\right)^{\frac{2}{3}}$;

then by (146) will $x\sqrt[3]{a+x} + a\sqrt[3]{a+x} = \frac{a}{b^{\frac{4}{3}}}x^{\frac{5}{3}}$:

$$\text{that is, } (a+x)\sqrt[3]{a+x} = \frac{a}{b^{\frac{4}{3}}}x^{\frac{5}{3}},$$

$$\text{or } (a+x)^{\frac{4}{3}} = \frac{a}{b^{\frac{4}{3}}}x^{\frac{5}{3}};$$

$\therefore$ cubing both members, we obtain $(a+x)^4 = \frac{a^3}{b^4}x^5$,

$$\text{or } a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4 = \frac{a^3}{b^4}x^5:$$

whence we have immediately

$$\frac{a^3}{b^4}x^5 - x^4 - 4ax^3 - 6a^2x^2 - 4a^3x = a^4,$$

$$\text{or } a^3x^5 - b^4x^4 - 4ab^4x^3 - 6a^2b^4x^2 - 4a^3b^4x - a^4b^4 = 0;$$

which is arranged according to the dimensions of $x$, and is free from irrational quantities.

148. By the proper application of the three propositions recited and exemplified in the last three articles, every equation however complicated, may be reduced so as to involve only integral and positive powers of the unknown quantity: and if the coefficient of its highest power be not unity, the equal division of both the members of the equation by that coefficient will in all cases reduce it to one or other of the following forms:

(1) $\quad x-p=0$, or $x=p$:

(2) $\quad x^2-px+q=0$, or $x^2-px=-q$:

(3) $\quad x^3-px^2+qx-r=0$, or $x^3-px^2+qx=r$:

&c.........................&c.................

(m) $\quad x^m-px^{m-1}+qx^{m-2}-\text{\&c.}\pm l=0$, or

$$x^m-px^{m-1}+qx^{m-2}-\text{\&c.}=\mp l;$$

wherein the known quantities $p$, $q$, $r$, &c. $l$ may be either positive or negative, integral, fractional or irrational.

The first is styled a *simple* equation, or equation of the *first order:* the second a *quadratic* equation, or equation of the *second order:* the third a *cubic* equation, or equation of the *third order*, &c. and the last an equation of *m dimensions*, or an equation of the $m^{\text{th}}$ *order.*

149. Equations of the kind we have been describing which may be either *numeral* or *literal*, or partly both, are sometimes termed *algebraical* by way of distinguishing them from such as have their members identical *independently* of any particular values of the unknown quantity and which are denominated *analytical;* as for instance $\sqrt{a^2+2ax+x^2}=a+x$, is an analytical or identical equation, the latter member being merely the result of the operation indicated in the former.

In the present Chapter our attention will be confined to the resolution of such equations only as belong to the first two classes above enumerated, or may be reduced to them by substitutions or other artifices; the general Theory of Equations being reserved for the Second Part of the work.

## I. SIMPLE EQUATIONS.

150. By means of articles (145), (146), (147) and (148), every equation coming under this head being capable of reduction to the form

$$x - p = 0, \quad \text{or } x = p,$$

it is obvious that its solution is thus effected, and that in each of the equations $x - p = 0$ and $x = p$, there is only one root or value of $x$ which satisfies the proposed condition; so that every simple equation has *one* and *only* one root.

This will amply appear in the treatment of the following equations, as also in the subsequent problems whose solutions are dependent upon simple equations.

Ex. 1. Given $4(x-3)+3x+1=2(x+2)$, to find the value of $x$.

Here by effecting the multiplications indicated, we have

$$4x - 12 + 3x + 1 = 2x + 4:$$

therefore, by transposition according to (145), we get

$$4x + 3x - 2x = 4 + 12 - 1, \text{ or } 5x = 15,$$

whence dividing both sides by 5, we obtain

$$x = \frac{15}{5} = 3:$$

that is, 3 is the value sought, which, being substituted in the place of $x$, will be found to render both sides of the equation identical.

Ex. 2. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} - \frac{x}{5} = x - 7$, to find the value of $x$.

Here by (63) the least common multiple of 2, 3, 4 and 5 being 60, we shall multiply both sides of the equation by 60 in order to clear it of fractions, and thence we get

$$30x + 20x + 15x - 12x = 60x - 420:$$

therefore by transposition

$$30x + 20x + 15x - 12x - 60x = -420, \text{ or } -7x = -420;$$

whence by equal division $x = \dfrac{-420}{-7} = 60$, the required value, which will be found to answer the proposed condition.

Ex. 3. Given $\dfrac{1}{x-2} - \dfrac{1}{x-4} = \dfrac{1}{x-6} - \dfrac{1}{x-8}$, to find $x$.

Reducing the terms on each side to a common denominator, and combining them respectively, we have

$$\frac{(x-4)-(x-2)}{(x-2)(x-4)} = \frac{(x-8)-(x-6)}{(x-6)(x-8)},$$

$$\text{or } -\frac{2}{(x-2)(x-4)} = -\frac{2}{(x-6)(x-8)};$$

$$\therefore \frac{1}{(x-2)(x-4)} = \frac{1}{(x-6)(x-8)},$$

$$\text{or } (x-6)(x-8) = (x-2)(x-4):$$

$$\text{that is, } x^2 - 14x + 48 = x^2 - 6x + 8,$$

whence, expunging $x^2$ from both sides, we have

$$48 - 8 = 14x - 6x, \text{ or } 40 = 8x, \text{ and } \therefore x = \frac{40}{8} = 5.$$

Ex. 4. Given $\dfrac{6x+7}{9} + \dfrac{7x+13}{6x+3} = \dfrac{2x+4}{3}$, to find $x$.

In the first place, multiplying both sides by 9 the least common multiple of 9 and 3, we have

$$6x + 7 + \frac{21x+39}{2x+1} = 6x + 12:$$

whence, rejecting $6x$ from both sides and transposing 7, we get

$$\frac{21x+39}{2x+1}=5,$$

$$\text{and } \therefore\ 21x+39=5\times(2x+1)=10x+5:$$

$$\therefore\ 21x-10x=5-39, \quad \text{or } 11x=-34,$$

$$\text{and } \therefore\ x=-\frac{34}{11}=-3\frac{1}{11}.$$

Ex. 5. Given $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}$, to find $x$.

As in the last example, multiplying every term by 36, the least common multiple of 3, 9, 12 and 36, we get

$$8x+34-\frac{468x-72}{17x-32}+12x=21x-x-16;$$

whence, by transposition and rejection of such quantities as are common to both sides, we have

$$50=\frac{468x-72}{17x-32}:$$

clearing this equation of fractions and transposing, we get

$$50(17x-32) \quad \text{or } 850x-1600=468x-72,$$

$$\text{and } \therefore\ 382x=1528,$$

from which, by the equal division of both sides by 382, we obtain

$$x=\frac{1528}{382}=4.$$

Ex. 6. Given

$$\frac{3ac}{a+b}+\frac{a^2b}{(a+b)^3}+\frac{(2a+b)bx}{a(a+b)^2}=\frac{3cx}{b}+\frac{x}{a}, \text{ to find } x.$$

Here, by transposition, we have

$$\frac{3ac}{a+b}+\frac{a^2b}{(a+b)^3}=\frac{3cx}{b}+\frac{x}{a}-\frac{(2a+b)bx}{a(a+b)^2},$$

and reducing the terms on each side to a common denominator, we get

$$\frac{3ac(a+b)^2+a^2b}{(a+b)^3}=\left\{\frac{3ac(a+b)^2+b(a+b)^2-(2a+b)b^2}{ab(a+b)^2}\right\}x$$

$$=\left\{\frac{3ac(a+b)^2+a^2b}{ab(a+b)^2}\right\}x;$$

$\therefore$ by equal division of both sides, we obtain

$$\frac{1}{(a+b)}=\frac{x}{ab},\text{ or } x=\frac{ab}{a+b} \text{ the value required.}$$

Ex. 7. Given $\sqrt{x+12}=6-\sqrt{x}$, to find $x$.

Squaring both sides, we get $x+12=36-12\sqrt{x}+x$, whence expunging $x$ which is found in both sides, and transposing,

we have $12\sqrt{x}=36-12=24$;

$\therefore \sqrt{x}=\frac{24}{12}=2$, and consequently $x=(\sqrt{x})^2=2^2=4$.

Ex. 8. Given $\sqrt{x}+\sqrt{x+3}=\frac{12}{\sqrt{x+3}}$, to find $x$.

Clearing of fractions we get $\sqrt{x^2+3x}+x+3=12$:

transposing we have $\sqrt{x^2+3x}=9-x$:

squaring both sides, $x^2+3x=81-18x+x^2$:

expunging $x^2$ and transposing, we obtain $21x=81$,

and $\therefore x=\frac{81}{21}=\frac{27}{7}=3\frac{6}{7}$.

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Ex. 9. Given $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}}=b$, to find $x$.

Clearing of fractions, we have

$$\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}=b\sqrt{a}+b\sqrt{a-\sqrt{a^2-ax}}:$$

$\therefore$ by transposition and division, we obtain

$$(1-b)\sqrt{a}=(1+b)\sqrt{a-\sqrt{a^2-ax}},$$

$$\text{and } \sqrt{a-\sqrt{a^2-ax}}=\left(\frac{1-b}{1+b}\right)\sqrt{a}:$$

squaring both sides and transposing, we get

$$\sqrt{a^2-ax}=a-\left(\frac{1-b}{1+b}\right)^2 a=\frac{4ab}{(1+b)^2};$$

and repeating the same process, we manifestly obtain

$$a^2-ax=\frac{16a^2b^2}{(1+b)^4}, \text{ and } \therefore\ x=a\left\{1-\frac{16b^2}{(1+b)^4}\right\}.$$

151. Cor. A great variety of Equations, which after reduction by the ordinary methods would belong to higher orders, may be solved by means of the articles above given for the resolution of simple equations: but it must be observed, that the solution will in general be incomplete, in consequence of factors which satisfy the equation by becoming equal to zero, being rejected from both its members. The following examples will illustrate this.

Ex. 1. Given $ax\sqrt{d^2x^2+d^2}+adx^2=bcx$, to find $x$.

Transposing and dividing every term by $x$, we have

$$a\sqrt{d^2x^2+d^2}=bc-adx:$$

squaring both sides, we get

$$a^2d^2x^2+a^2d^2=b^2c^2-2abcdx+a^2d^2x^2:$$

expunging $a^2 d^2 x^2$, and transposing, we have

$$\therefore\ 2abcdx = b^2c^2 - a^2d^2, \text{ and } \therefore\ x = \frac{b^2c^2 - a^2d^2}{2abcd}:$$

and it may be observed that 0, which as a value of $x$ manifestly satisfies the equation, has not been discovered in the solution.

Ex. 2. Given $x - a + \sqrt{a^2 + x\sqrt{4b^2 + x^2}} = 0$, to find $x$.

By transposition, $\sqrt{a^2 + x\sqrt{4b^2 + x^2}} = a - x$:

squaring both sides, $a^2 + x\sqrt{4b^2 + x^2} = a^2 - 2ax + x^2$:

expunging $a^2$ and dividing by $x$, $\sqrt{4b^2 + x^2} = x - 2a$:

squaring both sides, $4b^2 + x^2 = x^2 - 4ax + 4a^2$:

expunging $x^2$ and transposing, $4ax = 4(a^2 - b^2)$, whence we readily obtain

$$x = \frac{a^2 - b^2}{a}:$$

and here it may be remarked that by the division of both sides by $x$, one value 0 of $x$, which obviously satisfies the proposed condition, has been entirely lost sight of.

Ex. 3. Given

$$\frac{a + \sqrt{x}}{\sqrt{a} + \sqrt{a + \sqrt{x}}} + \frac{a - \sqrt{x}}{\sqrt{a} + \sqrt{a - \sqrt{x}}} = \sqrt{a},$$

to find $x$.

By transposition, we have

$$\frac{a + \sqrt{x}}{\sqrt{a} + \sqrt{a + \sqrt{x}}} = \sqrt{a} - \frac{a - \sqrt{x}}{\sqrt{a} + \sqrt{a - \sqrt{x}}}$$

$$= \frac{\sqrt{a}\sqrt{a - \sqrt{x}} + \sqrt{x}}{\sqrt{a} + \sqrt{a - \sqrt{x}}};$$

$\therefore$ by clearing of fractions, &c. we obtain

$$a\sqrt{a}+\sqrt{x}\sqrt{a-\sqrt{x}}=\sqrt{x}\sqrt{a+\sqrt{x}}+\sqrt{a}\sqrt{a^2-x};$$

whence $\sqrt{a}(a-\sqrt{a^2-x})=\sqrt{x}\{\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}}\}$,

that is

$$\frac{\sqrt{a}}{2}\{\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}}\}^2$$

$$=\sqrt{x}\{\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}}\}:$$

$\therefore$ multiplying by 2 and dividing both sides by the factor

$$\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}},$$

we get $\sqrt{a}\{\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}}\}=2\sqrt{x};$

$\therefore\ a^2-a\sqrt{a^2-x}=2x$, and $a^2-2x=a\sqrt{a^2-x};$

$\therefore\ a^4-4a^2x+4x^2=a^4-a^2x$; whence, $4x^2=3a^2x$,

$$\text{and } \therefore\ x=\frac{3}{4}a^2:$$

and in this instance, the value 0 of $x$ which also satisfies the equation has been passed over by reason of the equal division by the factors

$$\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}} \text{ and } x.$$

Ex. 4. Given $\sqrt[5]{\sqrt{x}+a}-\sqrt[5]{\sqrt{x}-a}=\sqrt[5]{2a}$, to find $x$.

First, by raising both members of the equation to the fifth power, we have

$$x^{\frac{1}{2}}+a-5(x^{\frac{1}{2}}+a)^{\frac{4}{5}}(x^{\frac{1}{2}}-a)^{\frac{1}{5}}+10(x^{\frac{1}{2}}+a)^{\frac{3}{5}}(x^{\frac{1}{2}}-a)^{\frac{2}{5}}$$

$$-10(x^{\frac{1}{2}}+a)^{\frac{2}{5}}(x^{\frac{1}{2}}-a)^{\frac{3}{5}}+5(x^{\frac{1}{2}}+a)^{\frac{1}{5}}(x^{\frac{1}{2}}-a)^{\frac{4}{5}}-x^{\frac{1}{2}}+a=2a:$$

$\therefore$ by expunging the quantities common to both sides, and changing the signs &c. we get

$$5(x^{\frac{1}{2}}+a)^{\frac{1}{5}}(x^{\frac{1}{2}}-a)^{\frac{1}{5}}\{(x^{\frac{1}{2}}+a)^{\frac{3}{5}}-(x^{\frac{1}{2}}-a)^{\frac{3}{5}}\}$$

$$=10(x^{\frac{1}{2}}+a)^{\frac{2}{5}}(x^{\frac{1}{2}}-a)^{\frac{2}{5}}\{(x^{\frac{1}{2}}+a)^{\frac{1}{5}}-(x^{\frac{1}{2}}-a)^{\frac{1}{5}}\}:$$

whence dividing every term by $5\ (x-a^2)^{\frac{1}{5}}$, we obtain

$$(x^{\frac{1}{2}}+a)^{\frac{3}{5}}-(x^{\frac{1}{2}}-a)^{\frac{3}{5}}=2\,(x-a^2)^{\frac{1}{5}}\{(x^{\frac{1}{2}}+a)^{\frac{1}{5}}-(x^{\frac{1}{2}}-a)^{\frac{1}{5}}\}:$$

but since $(x^{\frac{1}{2}}+a)^{\frac{1}{5}}-(x^{\frac{1}{2}}-a)^{\frac{1}{5}}=\sqrt[5]{2a}$, we have

$$(x^{\frac{1}{2}}+a)^{\frac{3}{5}}-(x^{\frac{1}{2}}-a)^{\frac{3}{5}}-3\,(x-a^2)^{\frac{1}{5}}\{(x^{\frac{1}{2}}+a)^{\frac{1}{5}}-(x^{\frac{1}{2}}-a)^{\frac{1}{5}}\}$$

$$=(2a)^{\frac{3}{5}},\text{ and }\therefore\ (x^{\frac{1}{2}}+a)^{\frac{3}{5}}-(x^{\frac{1}{2}}-a)^{\frac{3}{5}}$$

$$=(2a)^{\frac{3}{5}}+3\,(2a)^{\frac{1}{5}}(x-a^2)^{\frac{1}{5}}:$$

wherefore by substitution, there arises

$$(2a)^{\frac{3}{5}}+3\,(2a)^{\frac{1}{5}}(x-a^2)^{\frac{1}{5}}=2\,(2a)^{\frac{1}{5}}(x-a^2)^{\frac{1}{5}}:$$

$$\text{that is, }(x-a^2)^{\frac{1}{5}}=-(2a)^{\frac{2}{5}}:$$

$$\therefore\ x-a^2=-(2a)^2=-4a^2\text{ and }x=-3a^2.$$

If $x=a^2$ and $\therefore \sqrt{x}=\pm a$, it is obvious that the equation would be satisfied, but we have not met with this value in the solution above given, in consequence of the equal division of the terms by the factor $5\,(x-a^2)^{\frac{1}{5}}$ which being put $=0$, gives $x=a^2$.

*Problems dependent upon Simple Equations.*

152. Whenever a problem involving only one independent magnitude to be determined, is proposed to be solved algebraically, it will be necessary to assume one of the latter letters $x$, $y$, $z$,

&c. of the alphabet to represent that magnitude, and if need be, a requisite number of the former letters $a$, $b$, $c$, &c. to designate such magnitudes as are considered known or given: then to endeavour to translate the specified conditions into algebraical language, the different parts being connected together by means of the signs explained in Chap. I. according to the circumstances under which it is presented for solution: that is, by proceeding as if the quantity required were already determined, and it were desired to try whether it would answer the proposed conditions or not: and this being effected, the result will be an equation, the resolution of which, by means of the principles above illustrated, will give the value of the letter assumed for the required quantity expressed in terms of such as are supposed known.

As no specific general rules applicable to the infinite variety of problems that may occur, can be laid down, the reader must be content with such directions upon the subject as he may be enabled to collect from particular examples.

Ex. 1. Required a number whose fourth part exceeds its fifth part by 2.

Let $x$ be taken to represent the number required, then will $\frac{x}{4}$ and $\frac{x}{5}$ be adequate representations of its fourth and fifth parts: and the condition being that the former shall exceed the latter by 2, we shall have

$$\frac{x}{4} - \frac{x}{5} = 2,$$

an equation exhibiting the problem algebraically;

$\therefore$ by clearing of fractions we get

$$5x - 4x = 40,$$

or $x = 40$, the number sought, which obviously possesses the property specified, since $\frac{40}{4} - \frac{40}{5}$, or $10 - 8 = 2$.

Ex. 2. Let it be required to divide the number 21 into two parts, such that ten times one of them may exceed nine times the other by 1.

Assume $x$ to represent one of the parts required, then will $21-x$ obviously denote the other part:

$$\text{also } 10x \text{ and } 189-9x$$

will manifestly be adequate representations of ten times the former and nine times the latter:

whence, connecting these quantities according to the proposed condition, we shall have

$$10x-(189-9x)=1,$$

which is therefore the translation of the problem into algebraical language exhibited as an equation;

$$\therefore\ 10x+9x=1+189, \text{ or } 19x=190,$$

whence we have immediately

$$x=\frac{190}{19}=10, \text{ one of the parts,}$$

$$\text{and } \therefore\ 21-x=21-10=11, \text{ the other part:}$$

and it is readily seen that the numbers 10 and 11 will fulfil what is specified in the enunciation of the problem.

Ex. 3. Required two numbers whose sum is 60, and difference 10.

Here, if $x$ be assumed to represent the less of the two numbers, it is clear that the greater will be represented by $x+10$: also their sum is equivalent to

$$x+x+10, \text{ or } 2x+10:$$

but the problem requiring that this sum should be 60, we shall obviously have the equation

$$2x+10=60:$$

$$\text{whence } 2x=60-10=50,$$

$$\text{and } \therefore\ x=25,$$

so that the two numbers denoted by $x$ and $x+10$ are 25 and 35, which manifestly fulfil the conditions.

Ex. 4. Find two magnitudes whose sum shall be $16\frac{1}{2}$, and quotient 27.

Assuming $x$ to represent the less of the magnitudes, we shall have the greater $= 16\frac{1}{2} - x$, or $\dfrac{33-2x}{2}$:

whence, by the conditions of the problem, we must have

$$\frac{33-2x}{2x} = 27, \text{ and } \therefore\ 33-2x = 54x;$$

$$\therefore\ 33 = 56x, \text{ and } x = \frac{33}{56}, \text{ which is the less magnitude:}$$

$$\text{also } 16\tfrac{1}{2} - x = 16\tfrac{1}{2} - \frac{33}{56} = \frac{924}{56} - \frac{33}{56} = \frac{891}{56} = 15\frac{51}{56},$$

the greater magnitude; and these two quantities will be found to satisfy the conditions proposed.

Ex. 5. *A* possesses four times as much property as *B*, and their fortunes together amount to £10000: what is the property of each?

Assume $x = B$'s property; then it is obvious that $4x$ will on this hypothesis represent that of *A*: hence by the question we shall have $x+4x$, or $5x = 10000$, and $\therefore\ x = 2000$:

$$\therefore\ \left.\begin{array}{l} B\text{'s property} = x \ \ = 2000 \\ A\text{'s property} = 4x = 8000 \end{array}\right\} \text{ which together make £10000.}$$

Ex. 6. At an election 1071 votes were tendered, and the successful candidate came in by a majority of 147: required the number of votes in favour of each.

Let $x$ represent the number of votes of the losing candidate, then $x + 147$ will denote that of the successful candidate:

hence by the question $x + (x + 147) = 1071$, or $2x + 147 = 1071$:

$$\therefore\ 2x = 1071 - 147 = 924, \text{ and } x = 462:$$

wherefore the numbers were 462 and 609.

Ex. 7. A gentleman wishing to relieve a number of beggars, finds that if he give them 6*d.* a piece, he will have 20*d.* left, and that he has not enough by 14*d.* to allow of his giving them 8*d.* a piece: required the number of beggars and the money he possesses.

Let $x =$ the number of beggars; then it is obvious, from the two conditions of the problem, that both

$$6x + 20 \text{ and } 8x - 14$$

are adequate representations of his money in pence: therefore we must manifestly have

$$8x - 14 = 6x + 20,$$

$$\text{and } \therefore\ 2x = 34,$$

$$\text{whence } x = \frac{34}{2} = 17,$$

which denotes the number of beggars:

also the sum of money he had will be either

$$6x + 20 \text{ or } 8x - 14,$$

each of which by the substitution of 17 in the place of $x$ gives

$$122d. = 10s.\,2d.$$

Ex. 8. Three persons, $A$, $B$, $C$ are possessed of certain sums of money, such that $A$ and $B$ together have £120, $A$ and $C$ together have £140, and $B$ and $C$ together £150: what is the sum possessed by each?

Let $x = A$'s money; then, from the nature of the case,

$$120 - x = B\text{'s money},$$

$$\text{and } 140 - x = C\text{'s money}:$$

hence $B$ and $C$ together have

$$120 - x + 140 - x = 260 - 2x;$$

$\therefore$ by the problem, we have $260 - 2x = 150$;

whence $110 = 2x$, and $\therefore x = 55$;

that is, $A$'s money = £55.;

$\therefore$ $B$'s money $= 120 - x = 120 - 55 =$ £65.,

and $C$'s money $= 140 - x = 140 - 55 =$ £85.

Ex. 9. A father's age is triple that of his son, but at the end of 10 years it will be only double: what is the age of each?

Assume $x =$ the son's age;

$\therefore$ $3x =$ the father's age on the same hypothesis:

again, $3x + 10$ and $x + 10$ are the respective ages of the father and son at the end of 10 years:

hence by the condition of the problem, we have

$$3x + 10 = 2(x + 10) = 2x + 20;$$

$\therefore$ $x = 10$, so that the ages of the father and son represented by $3x$ and $x$ are 30 and 10 years respectively.

Ex. 10. $A$ and $B$ play together for a stake of 12$s$.: if $A$ win, he will be thrice as rich as $B$; but if he lose, he will be only twice as rich: how much money does each possess at first?

Let $x$ denote $A$'s property at first;

$\therefore$ $\dfrac{x+12}{3} =$ $B$'s property after losing 12$s$. to $A$;

whence $\dfrac{x+12}{3} + 12 = \dfrac{x+48}{3} =$ $B$'s property at first:

also $x - 12 = A$'s property after losing 12*s.*;

and $\dfrac{x+48}{3} + 12 = \dfrac{x+84}{3} = B$'s property after winning 12*s.*:

wherefore we shall have, by the condition of the question,

$$x - 12 = 2\left(\frac{x+84}{3}\right), \text{ or } 3x - 36 = 2x + 168;$$

$$\therefore\ x = 168 + 36 = 204s. = A\text{'s property at first,}$$

and $\dfrac{x+48}{3} = \dfrac{204+48}{3} = \dfrac{252}{3} = 84s. = B$'s property at first.

Ex. 11. An egg-merchant meeting with three customers, sells to the first of them half of his stock and one egg more: to the second he disposes of half the remainder and two eggs more: and to the third half of what he then had left and three eggs more; and he afterwards discovers that he has parted with his whole stock: what number of eggs had he at first?

Let $x$ = number he had first;

$$\therefore\ \frac{x}{2} + 1 = \text{number sold to the first customer,}$$

$$\text{and } \therefore\ \frac{x}{2} - 1 = \text{number then left:}$$

$$\text{also, } \frac{x}{4} - \frac{1}{2} + 2 = \text{number sold to the second customer,}$$

$$\text{and } \therefore\ \frac{x}{4} - \frac{1}{2} - 2 = \text{number then remaining:}$$

$$\text{again, } \frac{x}{8} - \frac{1}{4} - 1 + 3 = \text{number sold to the third customer,}$$

$$\text{and } \therefore\ \frac{x}{8} - \frac{1}{4} - 1 - 3 = 0, \text{ by the question:}$$

$$\text{whence } x - 2 - 8 - 24 = 0,$$

$$\therefore \; x = 34, \text{ the required number,}$$

which is readily shewn to answer the proposed conditions: for he sells to the first customer 18 and has then 16 left: to the second he sells 10 and has afterwards 6 left, which he sells to the third.

Ex. 12. A person disposes of turkeys at as many shillings each as the number he has, and returning back 1*s.* finds that if he had had one more to sell on the same condition and had returned back 2*s.*, he would have received 20*s.* more from his bargain: what number did he dispose of?

Let $x =$ the required number;

$$\therefore \; x^2 - 1 = \text{the number of shillings received:}$$

also, on the second hypothesis, it is manifest that

$(x+1)^2 - 2 =$ the number of shillings he would have received: whence by the condition of the problem, we get

$$(x+1)^2 - 2 = x^2 - 1 + 20, \text{ or } x^2 + 2x - 1 = x^2 + 19;$$

$\therefore \; 2x = 20$, and $x = 10$, the number sought: and the problem is very easily verified.

Ex. 13. A gentleman bequeaths his property as follows: to his eldest child he leaves £1800., and one sixth of the rest of his property; to the second twice that sum and one sixth of what then remained: to the third three times the same sum and one sixth of the remainder, and so on: and by this arrangement his property is divided equally among his children: how many were there and what was their fortune?

Let $x =$ the property bequeathed;

$$\therefore \; 1800 + \frac{x - 1800}{6} = 1500 + \frac{x}{6} = \text{fortune of the eldest,}$$

and $x - 1500 - \frac{x}{6} = \frac{5x}{6} - 1500 = $ sum remaining:

again, $3600 + \frac{1}{6}\left\{\frac{5x}{6} - 1500 - 3600\right\} = 2750 + \frac{5x}{36} = $ fortune of the second: whence by the question, we have

$$1500 + \frac{x}{6} = 2750 + \frac{5x}{36}, \text{ and } \therefore x = \pounds 45000.,$$

the property bequeathed:

also, $1500 + \frac{x}{6} = 1500 + 7500 = \pounds 9000.$, the fortune of each;

and $\therefore \frac{45000}{9000} = 5$, the number of children: and the correctness of the solution is capable of very easy proof.

Ex. 14. A pack of $p$ cards is distributed into $n$ heaps, so that the number of pips on the lowest cards together with the number of cards laid upon them, is the same given number $m$ for each heap, and the number of cards then remaining is found to be $r$: required the number of pips on all the lowest cards.

Let $x =$ the number of pips on all the lowest cards;

then, since $mn =$ the number of pips together with the number of cards laid upon the lowest, we shall have

$mn - x =$ the number of cards laid upon all the lowest:

$\therefore mn - x + n =$ the number of cards in all the heaps:

whence we shall manifestly have $mn - x + n + r = p$,

and $\therefore x = mn + n + r - p = (m+1)n + r - p$,

the required number.

This trick may be readily performed by means of a common pack of cards.

Ex. 15. *A* and *B* are possessed of certain sums of money such that if they gain £$a$ and £$b$ respectively, *A* will be $m$ times as rich as *B*; but if they gain £$c$ and £$d$ respectively, *A* becomes possessed of $n$ times as much as *B*: required the money of each.

Let $x = A$'s money at first;

$$\therefore \frac{x+a}{m} = B\text{'s money after the first gain:}$$

$$\therefore \frac{x+a}{m} - b = B\text{'s money at first:}$$

$$\text{again } x + c = A\text{'s money after the second gain,}$$

$$\text{and } \frac{x+a}{m} - b + d = B\text{'s money after the second gain:}$$

whence we have by the question

$$x + c = n\left(\frac{x+a}{m} - b + d\right),$$

$$\text{or} \quad mx + mc = nx + na - mn(b-d):$$

$$\therefore (m-n)x = na - mnb + mnd - mc$$

$$= m(nd-c) - n(mb-a),$$

$$\text{and } x = \frac{m(nd-c) - n(mb-a)}{m-n} = A\text{'s money at first;}$$

$$\therefore \frac{x+a}{m} - b = \frac{(nd-c)-(mb-a)}{m-n} = B\text{'s money at first.}$$

Before quitting this subject, we will examine some of the various modifications of the result which depend upon the relative magnitudes of the quantities $m$ and $n$.

(1). If $m$ be greater than $n$, it is obvious that in order to satisfy the conditions as enunciated in the problem, we must have

$$m(nd-c) > n(mb-a) \text{ and } (nd-c) > mb-a.$$

(2). If $m$ be less than $n$, then since

$$A\text{'s money} = \frac{m(nd-c)-n(mb-a)}{m-n} = \frac{n(mb-a)-m(nd-c)}{n-m},$$

$$\text{and } B\text{'s money} = \frac{(nd-c)-(mb-a)}{m-n} = \frac{(mb-a)-(nd-c)}{n-m},$$

we must obviously have

$$n(mb-a) > m(nd-c) \text{ and } mb-a > nd-c.$$

(3). If $m = n$, it will follow from the same mode of reasoning, that we must have

$$m(md-c) = m(mb-a) \text{ and therefore } md-c = mb-a,$$

from which there obviously results

$$md-c = mb-a, \text{ or } a-c = m(b-d);$$

and both $x$ and $\frac{x+a}{m} - b$ assume the indeterminate form $\frac{0}{0}$.

In this case, we have

$$A\text{'s money} = m\left\{\frac{(md-c)-(mb-a)}{m-m}\right\},$$

$$B\text{'s money} = \left\{\frac{(md-c)-(mb-a)}{m-m}\right\};$$

which shew that $A$'s original sum is $m$ times as great as $B$'s, and the results $\frac{0}{0}$ indicate that any sums whatever, one of which is $m$ times as great as the other, will satisfy the proposed conditions, provided that $a-c$ is $m$ times as great as $b-d$.

(4). If $m$ be greater than, less than, or equal to $n$, but corresponding thereto $m(nd-c)$ and $nd-c$ be not greater than, less than, or equal to, $n(mb-a)$ and $mb-a$ respectively, the sums originally possessed by $A$ and $B$ become negative, and the problem is impossible as it is enunciated: but in order that the solution may be applicable in the case just mentioned, the enunciation of the problem must manifestly stand as follows:

$A$ and $B$ *owe* certain sums of money, such that if &c.

(5). If the expression for the sum originally possessed by $A$ were positive and for that by $B$ negative, the enunciation would be

$A$ and $B$ *possess* and *owe* respectively certain sums of money, such that if &c.

(6). If we change the signs of one or more of the quantities $a$, $b$, $c$, $d$ in the solution above given from $+$ to $-$, we must obviously change the corresponding word *gain* into the word *lose*, in order that the statement of the problem may be consistent with the result obtained.

## II. QUADRATIC EQUATIONS.

153. The equations belonging to this class being by means of the propositions laid down in the former part of the chapter reducible generally to the form

$$x^2-px+q=0 \text{ or } x^2-px=-q,$$

and, if the coefficient of the second term vanish, to the form

$$x^2+q=0 \text{ or } x^2=-q,$$

where $p$ and $q$ may be either positive or negative, integral, fractional or irrational; it now only remains to devise general methods by which the values of $x$ may be determined from any of these equations in terms of the known quantities $p$ and $q$.

Equations of the forms $x^2+q=0$ and $x^2=-q$ are termed *pure* quadratics, and those of the forms $x^2-px+q=0$ and $x^2-px=-q$ are called *adfected* quadratics.

*Pure Quadratics.*

154. In all equations which are reducible to the general form of pure quadratics

$$x^2+q=0 \text{ or } x^2=-q,$$

it is manifest that if the square roots of both sides of the latter form be extracted, we shall find

$$x=\pm\sqrt{-q}.$$

Consequently every pure quadratic, properly so called, has *two* and *only* two roots, which are equal in magnitude, but different in algebraical sign.

155. Cor. Hence, therefore, if $a$ and $-a$ denote the two roots, we shall obviously have $x^2+q=(x-a)(x+a)=0$, which will be satisfied if $x-a=0$ and $x+a=0$: and thus the equation may be solved by the decomposition of the former form into two factors and making each of them equal to zero.

Ex. 1. Given $8x^2-15=185$, to find the values of $x$.

By transposition, $8x^2=185+15=200$;

whence $x^2=25$, and $\therefore x=\pm 5$, each of which values satisfies the equation.

This equation, after reduction, is

$$x^2-25=(x-5)(x+5)=0.$$

Ex. 2. Given $\dfrac{x+7}{x^2-7x}-\dfrac{x-7}{x^2+7x}=\dfrac{7}{x^2-73}$, to find the values of $x$.

First, by reducing the former fractions, we have

$$\frac{x(x+7)^2-x(x-7)^2}{x^4-49x^2}=\frac{7}{x^2-73}, \text{ or } \frac{28}{x^2-49}=\frac{7}{x^2-73}:$$

and dividing both sides by 7, we obtain $\dfrac{4}{x^2-49}=\dfrac{1}{x^2-73}$;

$$\text{whence } 4x^2 - 292 = x^2 - 49,$$

$$\text{and } \therefore\ x^2 = 81 \text{ or } x = \pm 9:$$

and both these values will be found to fulfil the condition.

Ex. 3. Given $\sqrt{x+a} = \sqrt{x + \sqrt{b^2 + x^2}}$, to find the values of $x$.

Squaring both sides, we have $x + a = x + \sqrt{b^2 + x^2}$;

$$\therefore\ a = \sqrt{b^2 + x^2},$$

$$\text{whence } a^2 = b^2 + x^2, \text{ and } x^2 = a^2 - b^2;$$

$$\therefore\ x = \pm\sqrt{a^2 - b^2}$$

both of which may easily be proved to satisfy the equation.

156. All equations reducible to the binomial forms

$$x^m + q = 0 \text{ or } x^m = -q,$$

may be solved by the same method: and the values of $x$ are expressed by the quantities comprised in $\sqrt[m]{-q}$, which will manifestly have the same sign as $q$ when $m$ is odd, and both signs when $m$ is even.

Ex. 1. Given $a = \sqrt[3]{x^3 + \sqrt{x^6 - a^6}}$, to find the value of $x$.

Cubing both sides, $a^3 = x^3 + \sqrt{x^6 - a^6}$;

$$\therefore\ a^3 - x^3 = \sqrt{x^6 - a^6}:$$

squaring both sides, $a^6 - 2a^3x^3 + x^6 = x^6 - a^6$;

$$\therefore\ 2a^3x^3 = 2a^6 \text{ and } x^3 = a^3,$$

whence we have $x = \sqrt[3]{a^3}$ the only possible value of which is $a$.

Ex. 2. Let $\dfrac{a^2 + \sqrt[n]{a^{2n} - x^{2n}}}{a^2 - \sqrt[n]{a^{2n} - x^{2n}}} = \dfrac{b}{c}$, then we have

$$ca^2 + c\sqrt[n]{a^{2n} - x^{2n}} = ba^2 - b\sqrt[n]{a^{2n} - x^{2n}}:$$

$$\therefore\ (b+c)\sqrt[n]{a^{2n}-x^{2n}}=(b-c)\,a^2,$$

$$\text{whence } \sqrt[n]{a^{2n}-x^{2n}}=\left(\frac{b-c}{b+c}\right)a^2:$$

$$\therefore \text{ by involution, } a^{2n}-x^{2n}=\left(\frac{b-c}{b+c}\right)^n a^{2n},$$

$$\text{and } x^{2n}=a^{2n}-\left(\frac{b-c}{b+c}\right)^n a^{2n}=a^{2n}\left\{1-\left(\frac{b-c}{b+c}\right)^n\right\}:$$

$$\text{consequently } x=\pm a\sqrt[2n]{1-\left(\frac{b-c}{b+c}\right)^n}.$$

### *Adfected Quadratics.*

157. In every equation reducible to the general form of adfected quadratics $x^2-px=-q$,

$$\because\ \left(x-\frac{p}{2}\right)^2=x^2-px+\frac{p^2}{4},$$

it is obvious that if to both its members, the quantity $\frac{p^2}{4}$ or *the Square of half the Coefficient of the second Term* be added, the former side becomes a complete square, and the latter consists of known quantities, so that

$$\left(x-\frac{p}{2}\right)^2=\frac{p^2}{4}-q;$$

therefore by extracting the square roots of both sides, we obtain

$$x-\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}-q},$$

$$\text{whence } x=\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}=\frac{p\pm\sqrt{p^2-4q}}{2}:$$

and the two values $\frac{p+\sqrt{p^2-4q}}{2}$ and $\frac{p-\sqrt{p^2-4q}}{2}$ of $x$ will be found to satisfy the equation proposed.

Hence every equation properly termed an adfected quadratic, has *two* and *only* two roots, which may be found by what is called *completing the Square* as above explained.

158. Cor. 1. Since, by actual multiplication,

$$\left\{x-\left(\frac{p+\sqrt{p^2-4q}}{2}\right)\right\}\left\{x-\left(\frac{p-\sqrt{p^2-4q}}{2}\right)\right\}$$

$$=\left\{\left(x-\frac{p}{2}\right)-\frac{\sqrt{p^2-4q}}{2}\right\}\left\{\left(x-\frac{p}{2}\right)+\frac{\sqrt{p^2-4q}}{2}\right\}$$

$$=\left(x-\frac{p}{2}\right)^2-\left(\frac{p^2-4q}{4}\right)=x^2-px+q,$$

if we suppose the two values $\frac{p+\sqrt{p^2-4q}}{2}$ and $\frac{p-\sqrt{p^2-4q}}{2}$ of $x$ to be denoted by $a$ and $b$, we shall obviously have

$$(x-a)(x-b)=x^2-px+q=0;$$

and thence we conclude that every quadratic equation as

$$x^2-px+q=0$$

is resolvable into two simple equations, since its conditions are satisfied by making both

$$x-a=0 \text{ and } x-b=0.$$

Also, since $a+b=\frac{p+\sqrt{p^2-4q}}{2}+\frac{p-\sqrt{p^2-4q}}{2}=p,$

and $ab=\frac{p+\sqrt{p^2-4q}}{2}\times\frac{p-\sqrt{p^2-4q}}{2}=q:$

that is,

$$-p=-(a+b) \text{ and } q=(-a)\times(-b),$$

it follows that the coefficients of the second and third terms

of any quadratic equation, reduced to its proper form, are respectively equal to the sum and product of the roots with their signs changed.

159. Cor. 2. It follows, therefore, that if the former member of an equation of two dimensions can be decomposed into two simple factors, the roots of the quadratic may be found by the resolution of the two simple equations arising from equating each of these factors to zero.

160. Cor. 3. If $p^2=4q$ or $\sqrt{p^2-4q}=0$, each of the roots is $\frac{p}{2}$ and they are equal to one another: but if the quantity $\sqrt{p^2-4q}$ be either irrational or imaginary, it is evident that both the roots of the quadratic will be either irrational or imaginary: in other words, if the coefficients of a quadratic equation be rational, surd and impossible roots always occur in pairs.

161. Cor. 4. Hence, also, if we wish to resolve any quantity of two dimensions as $x^2-px+q$ into its constituent simple factors, we have only to find the roots $a$ and $b$ of the equation

$$x^2-px+q=0,$$

for then will

$$x^2-px+q=(x-a)(x-b).$$

Ex. 1. Given $x^2-6x+12=4$, to find the values of $x$.

Here transposing, and then completing the square by adding to both sides the square of half the coefficient of the second term, we have

$$x^2-6x+9=-8+9=1:$$

whence extracting the square roots of both sides, we get

$$x-3=\pm 1$$

$$\text{and } \therefore\ x=\pm 1+3=4 \text{ and } 2:$$

and on trial it will be found that both 4 and 2 satisfy the proposed equation.

Also, after reduction to the form $x^2-6x+8=0$, the equation is equivalent to $(x-2)(x-4)=0$.

Ex. 2. Given $x^2+7x+2=10$, to find the values of $x$.

By transposition we have $x^2+7x=8$;

completing the square, we get $x^2+7x+\left(\frac{7}{2}\right)^2=8+\left(\frac{7}{2}\right)^2=\frac{81}{4}$;

extracting the square roots, we obtain $x+\frac{7}{2}=\pm\frac{9}{2}$:

and by transposition $x=\pm\frac{9}{2}-\frac{7}{2}=1$ and $-8$, the two roots.

Hence we have the trinomial $x^2+7x-8=(x-1)(x+8)$.

Ex. 3. Given $x^2-\left(\frac{x-5}{2}\right)^2-7x-35=0$, to find the values of $x$.

Clearing of fractions, $4x^2-x^2+10x-25-28x-140=0$;

$$\therefore\ 3x^2-18x=165, \text{ and } x^2-6x=55:$$

whence, completing the square, we have

$$x^2-6x+9=64:$$

$\therefore$ extracting the square roots of both sides, we get

$$x-3=\pm 8, \text{ and } \therefore\ x=\pm 8+3=11 \text{ and } -5,$$

which are the required roots.

Ex. 4. Given $acx^2-bcx+adx=bd$, to find the values of $x$.

First, dividing every term by $ac$ in order to reduce the equation to the proper form, we have

$$x^2-\left(\frac{b}{a}-\frac{d}{c}\right)x=\frac{bd}{ac}:$$

hence, completing the square, we then obtain

$$x^2-\left(\frac{b}{a}-\frac{d}{c}\right)x+\frac{1}{4}\left(\frac{b}{a}-\frac{d}{c}\right)^2=\frac{bd}{ac}+\frac{1}{4}\left(\frac{b}{a}-\frac{d}{c}\right)^2=\frac{1}{4}\left(\frac{b}{a}+\frac{d}{c}\right)^2:$$

$\therefore$ by extraction of the square roots of both sides, there results

$$x-\frac{1}{2}\left(\frac{b}{a}-\frac{d}{c}\right)=\pm\frac{1}{2}\left(\frac{b}{a}+\frac{d}{c}\right),$$

and thence $x=\pm\frac{1}{2}\left\{\frac{b}{a}+\frac{d}{c}\right\}+\frac{1}{2}\left\{\frac{b}{a}-\frac{d}{c}\right\}=\frac{b}{a}$ and $-\frac{d}{c}$,

the two roots, which will both be found to satisfy the equation.

Ex. 5. Given $\sqrt{\dfrac{4+\sqrt{x^2+2x^3}}{x}}=\dfrac{\sqrt{x}}{2}+\dfrac{2}{\sqrt{x}}$, to find the values of $x$.

Squaring both members, we have

$$\frac{4+\sqrt{x^2+2x^3}}{x}=\frac{x}{4}+2+\frac{4}{x}:$$

expunging $\frac{4}{x}$ from both sides, and reducing the former,

$$\sqrt{1+2x}=\frac{x}{4}+2:$$

squaring both sides again, $1+2x=\dfrac{x^2}{16}+x+4:$

whence $x^2-16x=-48:$

completing the square, $x^2-16x+64=16$, wherefore

$x-8=\pm 4$ and $x=12$ and $4$,

each of which will upon trial be found to satisfy the equation.

Ex. 6. Given $\sqrt{10x+11}=44-5x$, to find the values of $x$.

Squaring both members, $10x+11=1936-440x+25x^2$;

$$\therefore\ 25x^2-450x=-1925 \text{ by transposition,}$$

$$\text{and } x^2-18x=-77:$$

$$\text{completing the square, } x^2-18x+81=-77+81=4,$$

$$\text{extracting the roots, } x-9=\pm 2;$$

$$\therefore\ x=\pm 2+9=11 \text{ and } 7;$$

the latter of which alone answers the condition of the equation: and the former value of $x$ corresponds to the equation

$$-\sqrt{10x+11}=44-5x \text{ or } \sqrt{10x+11}=5x-44,$$

because its solution is equally comprised in the operation above given in consequence of the radical quantity admitting of the double sign $\pm$.

Ex. 7. Given $2x+\sqrt{x^2-7}=5$, to find the values of $x$.

Here we have $\sqrt{x^2-7}=5-2x$;

$$\therefore \text{ by involution, } x^2-7=25-20x+4x^2:$$

$$\text{whence } 3x^2-20x=-32 \text{ and } x^2-\frac{20}{3}x=-\frac{32}{3}:$$

$$\therefore\ x^2-\frac{20}{3}x+\frac{100}{9}=-\frac{32}{3}+\frac{100}{9}=\frac{4}{9},$$

$$\therefore \text{ by evolution, } x-\frac{10}{3}=\pm\frac{2}{3};$$

$$\text{and } x=\pm\frac{2}{3}+\frac{10}{3}=4 \text{ and } \frac{8}{3}:$$

but in this instance, it will appear upon trial that neither 4, $\frac{8}{3}$ nor any other quantity that can be found will satisfy the equation: in fact these two quantities are the roots of the equation $2x - \sqrt{x^2 - 7} = 5$, whose solution, by reason of the double sign of the radical quantity, is comprehended in that of the one proposed.

Ex. 8. Given $x + \sqrt{bx + a^2} = a$, to find the values of $x$.

First, $\sqrt{bx + a^2} = a - x$;

$$\therefore\ bx + a^2 = a^2 - 2ax + x^2,$$

$$\text{and } x^2 - (2a + b)\,x = 0:$$

$$\text{whence } x^2 - (2a+b)x + \frac{(2a+b)^2}{4} = \frac{(2a+b)^2}{4};$$

$$\therefore\ x - \frac{2a+b}{2} = \pm\frac{2a+b}{2},$$

$$\text{and } x = \pm\frac{2a+b}{2} + \frac{2a+b}{2} = 2a + b \text{ and } 0,$$

which will both be found to fulfil the condition.

This equation, after reduction to a proper form, may be written

$$\{x - (2a + b)\}\,x = 0;$$

whence we have immediately

$$x - (2a + b) = 0 \text{ and } x = 0,$$

$$\text{or } x = 2a + b \text{ and } x = 0 \text{ as before:}$$

but if we had divided both its members by $x$, we should have entirely lost sight of the root 0, which satisfies the equation as well as $2a + b$.

Ex. 9. Given

$$2\sqrt{\frac{x - 3a}{3}} + \sqrt{6x} = \frac{21a + 5x}{\sqrt{3x - 9a}}, \text{ to find } x.$$

Clearing of fractions,

$$2x-6a+\sqrt{18x^2-54ax}=21a+5x;$$

by transposition, $\sqrt{18x^2-54ax}=27a+3x$;

clearing of surds, $18x^2-54ax=729a^2+162ax+9x^2$;

by transposition, $9x^2-216ax=729a^2$;

$\therefore$ by division, we get $x^2-24ax=81a^2$:

$\therefore$ completing the square and extracting the square roots, we obtain $x^2-24ax+144a^2=225a^2$, and $x-12a=\pm 15a$;

whence $x=\pm 15a+12a=27a$ and $-3a$.

Ex. 10. Given

$$\sqrt{\frac{3x}{x+1}}-\sqrt{\frac{x+1}{3x}}=2, \text{ to find the values of } x.$$

Squaring both sides, we have $\dfrac{3x}{x+1}-2+\dfrac{x+1}{3x}=4$;

by transposition, $\dfrac{3x}{x+1}+\dfrac{x+1}{3x}=6$;

clearing of fractions, $9x^2+x^2+2x+1=18x^2+18x$;

$\therefore$ by transposition and division, $x^2+2x=\dfrac{1}{8}$:

completing the square, $x^2+2x+1=1+\dfrac{1}{8}=\dfrac{9}{8}$:

whence, extracting the square roots, we obtain

$$x+1=\pm\frac{3}{2\sqrt{2}}, \text{ and } \therefore x=-1\pm\frac{3}{2\sqrt{2}}.$$

162. If an equation, after the requisite reductions are made, assume the form

$$x^{2m}-px^m+q=0 \text{ or } x^{2m}-px^m=-q,$$

where $m$ is either positive or negative, integral or fractional, it is obvious that the solution may be effected by the same process.

For, since $\left(x^m - \frac{p}{2}\right)^2 = x^{2m} - px^m + \frac{p^2}{4}$, we shall have

$$\left(x^m - \frac{p}{2}\right)^2 = \frac{p^2}{4} - q;$$

whence by evolution, $x^m - \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}$,

$$\therefore\ x^m = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} = \frac{p \pm \sqrt{p^2 - 4q}}{2},$$

and thence $x = \sqrt[m]{\dfrac{p \pm \sqrt{p^2 - 4q}}{2}}$.

Ex. 1. Given $x^6 - 6x^3 = 16$, to find the values of $x$.

Completing the square, $x^6 - 6x^3 + 9 = 16 + 9 = 25$;

extracting the square roots, $x^3 - 3 = \pm 5$;

whence $x^3 = \pm 5 + 3 = 8$ and $-2$,

and $\therefore\ x = 2$, and $\sqrt[3]{-2}$.

In this example, we have discovered only two roots, though the equation is of six dimensions; but since the cube of each of the quantities 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$, is equal to 1, it follows that the cube roots of 8 and $-2$ will each have three different values:

the former being 2, $-1+\sqrt{-3}$ and $-1-\sqrt{-3}$, and the latter

$$\sqrt[3]{-2},\ \frac{-1+\sqrt{-3}}{2}\sqrt[3]{-2} \text{ and } \frac{-1-\sqrt{-3}}{2}\sqrt[3]{-2};$$

all of which are therefore roots of the equation.

Ex. 2. Given $x^{-4} - 9x^{-2} + 20 = 0$, to find the values of $x$.

Completing the square, &c. $x^{-4} - 9x^{-2} + \frac{81}{4} = \frac{1}{4}$;

extracting the square roots, $x^{-2} - \frac{9}{2} = \pm\frac{1}{2}$;

$$\therefore\ x^{-2} = \pm\frac{1}{2} + \frac{9}{2} = 5 \text{ and } 4: \text{ that is } \frac{1}{x^2} = 5 \text{ and } 4,$$

$$\text{and } \therefore\ x = \pm\frac{1}{\sqrt{5}} \text{ and } \pm\frac{1}{2},$$

which are the roots required, and the same as would have been found by reducing the equation to the form $x^4 - \frac{9}{20}x^2 = -\frac{1}{20}$.

Ex. 3. Given $\sqrt[4]{x} + 7\sqrt{x} = 116$, to find the values of $x$.

Here $7x^{\frac{1}{2}} + x^{\frac{1}{4}} = 116$, and $\therefore\ x^{\frac{1}{2}} + \frac{1}{7}x^{\frac{1}{4}} = \frac{116}{7}$;

completing the square, $x^{\frac{1}{2}} + \frac{1}{7}x^{\frac{1}{4}} + \frac{1}{196} = \frac{3249}{196}$;

by evolution, $x^{\frac{1}{4}} + \frac{1}{14} = \pm\frac{57}{14}$, $\therefore\ x^{\frac{1}{4}} = 4$ and $-\frac{29}{7}$;

whence $x = 4^4 = 256$, and $\left(-\frac{29}{7}\right)^4 = \frac{707281}{2401}$.

If this equation had first been reduced to a rational form, the same values of $x$ would have been obtained.

163. Many other equations, which by the ordinary processes of reduction would rise to higher dimensions than what are properly comprised in this class, may be solved by completing the square, if they can be made to assume the form

$$y^2 - Py + Q = 0,$$

where $y$ involves $x$ or its powers and roots, combined with quantities that are known: but as no general directions can be given in addition to what have been already laid down, the reader must rely upon his own judgment in the choice of the method which he may adopt.

Ex. 1. Given $x^2 + \frac{1}{x^2} = \frac{35x^2 - 62x + 35}{6x}$, to find the values of $x$.

First, $x^2 + \frac{1}{x^2} = \frac{35}{6}x - \frac{31}{3} + \frac{35}{6x} = \frac{35}{6}\left(x + \frac{1}{x}\right) - \frac{31}{3}$,

$$\text{or } \left(x + \frac{1}{x}\right)^2 - \frac{35}{6}\left(x + \frac{1}{x}\right) = -\frac{25}{3}:$$

$\therefore$ by completing the square, considering $\left(x + \frac{1}{x}\right) = y$, we have

$$\left(x + \frac{1}{x}\right)^2 - \frac{35}{6}\left(x + \frac{1}{x}\right) + \left(\frac{35}{12}\right)^2 = \frac{25}{144};$$

$$\therefore\ x + \frac{1}{x} - \frac{35}{12} = \pm\frac{5}{12},$$

$$\text{and } x + \frac{1}{x} = \pm\frac{5}{12} + \frac{35}{12} = \frac{10}{3} \text{ and } \frac{5}{2}:$$

from the former, $x + \frac{1}{x} = \frac{10}{3}$;

$$\therefore\ x^2 - \frac{10}{3}x = -1 \text{ and } x^2 - \frac{10}{3}x + \frac{25}{9} = \frac{16}{9},$$

$$\text{whence } x - \frac{5}{3} = \pm\frac{4}{3}, \text{ and } \therefore x = 3 \text{ and } \frac{1}{3}:$$

from the latter, $x + \frac{1}{x} = \frac{5}{2}$;

$$\therefore\ x^2 - \frac{5}{2}x = -1 \text{ and } x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{9}{16},$$

$$\text{whence } x - \frac{5}{4} = \pm\frac{3}{4}, \text{ and } \therefore\ x = 2 \text{ and } \frac{1}{2}:$$

therefore the four values of $x$ which satisfy the equation proposed are $3$, $\frac{1}{3}$, $2$ and $\frac{1}{2}$.

Ex. 2. Given $2x^2 + \sqrt{2x^2 + 1} = 11$, to find the values of $x$.

Adding 1 to both sides of the equation, we have

$$(2x^2 + 1) + \sqrt{2x^2 + 1} = 12;$$

$\therefore$ considering $2x^2 + 1$ as a simple quantity and completing the square, we get

$$(2x^2 + 1) + \sqrt{2x^2 + 1} + \frac{1}{4} = \frac{49}{4};$$

$$\therefore \text{ by evolution, } \sqrt{2x^2 + 1} + \frac{1}{2} = \pm \frac{7}{2};$$

$$\text{whence } \sqrt{2x^2 + 1} = 3 \text{ and } -4;$$

$$\therefore 2x^2 + 1 = 9 \text{ and } 16, \; 2x^2 = 8 \text{ and } 15;$$

$$\therefore x^2 = 4 \text{ and } \frac{15}{2}, \text{ and } x = \pm 2 \text{ and } \pm \sqrt{\frac{15}{2}}.$$

Ex. 3. Given $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \dfrac{3x + 33}{2}$, to find the values of $x$.

Clearing of fractions, transposing and adding 6 to both sides, we have

$$(2x^2 - 5x + 6) + 10\sqrt{2x^2 - 5x + 6} = 39;$$

whence completing the square as before, we obtain

$$(2x^2 - 5x + 6) + 10\sqrt{2x^2 - 5x + 6} + 25 = 64;$$

$\therefore$ by evolution and transposition, $\sqrt{2x^2 - 5x + 6} = 3$ and $-13$; and clearing of surds, we find $2x^2 - 5x + 6 = 9$ and $169$:

$$\text{first, let } 2x^2 - 5x + 6 = 9, \quad \therefore x^2 - \frac{5}{2}x = \frac{3}{2},$$

$$\therefore x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{49}{16},$$

$$\therefore x - \frac{5}{4} = \pm \frac{7}{4}, \text{ whence } x = 3 \text{ and } -\frac{1}{2}:$$

next, let $2x^2-5x+6=169$, $\therefore\ x^2-\frac{5}{2}x=\frac{163}{2}$;

$$\therefore\ x^2-\frac{5}{2}x+\frac{25}{16}=\frac{1329}{16},$$

$$\therefore\ x-\frac{5}{4}=\pm\frac{\sqrt{1329}}{4} \text{ and } x=\frac{5\pm\sqrt{1329}}{4};$$

and all these four values of $x$ satisfy the equation.

Ex. 4. Given $\sqrt[4]{a+x}+\sqrt[4]{a-x}=\sqrt[4]{b}$, to find the values of $x$.

Squaring both sides, $\sqrt{a+x}+2\sqrt[4]{a^2-x^2}+\sqrt{a-x}=\sqrt{b}$;

transposing, $\sqrt{a+x}+\sqrt{a-x}=\sqrt{b}-2\sqrt[4]{a^2-x^2}$;

squaring both sides again, we get

$$2a+2\sqrt{a^2-x^2}=b-4\sqrt{b}\sqrt[4]{a^2-x^2}+4\sqrt{a^2-x^2};$$

$$\text{whence } \sqrt{a^2-x^2}-2\sqrt{b}\sqrt[4]{a^2-x^2}=a-\frac{b}{2};$$

completing the square, we have

$$\sqrt{a^2-x^2}-2\sqrt{b}\sqrt[4]{a^2-x^2}+b=a+\frac{b}{2};$$

$$\therefore \text{ by evolution, } \sqrt[4]{a^2-x^2}-\sqrt{b}=\pm\sqrt{a+\frac{b}{2}};$$

$$\text{and } \sqrt[4]{a^2-x^2}=\sqrt{b}\pm\sqrt{a+\frac{b}{2}};$$

$$\text{whence } a^2-x^2=\left\{\sqrt{b}\pm\sqrt{a+\frac{b}{2}}\right\}^4,$$

$$\text{and } \therefore\ x=\pm\sqrt{a^2-\left\{\sqrt{b}\pm\sqrt{a+\frac{b}{2}}\right\}^4}.$$

Ex. 5. Given $\sqrt[m]{(1+x)^2} - \sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}$, to find the values of $x$.

Dividing both members by $\sqrt[m]{(1-x)^2}$, we have

$$\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - 1 = \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}},$$

whence $\left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = 1$, which is of the specified form:

$$\therefore \left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} + \frac{1}{4} = \frac{5}{4},$$

and $\therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} - \frac{1}{2} = \frac{\pm\sqrt{5}}{2}$, or $\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = \frac{1\pm\sqrt{5}}{2}$:

$\therefore$ by involution, $\dfrac{1+x}{1-x} = \dfrac{(1\pm\sqrt{5})^m}{2^m}$,

and thence $2^m + 2^m x = (1\pm\sqrt{5})^m - (1\pm\sqrt{5})^m x$,

$$\therefore x = \frac{(1\pm\sqrt{5})^m - 2^m}{(1\pm\sqrt{5})^m + 2^m}.$$

Ex. 6. Given $\sqrt{2x+7} + \sqrt{3x-18} = \sqrt{7x+1}$, to find the values of $x$.

Assume $y = 2x+7$ and $\therefore x = \dfrac{y-7}{2}$, whence by substitution, the equation becomes

$$\sqrt{y} + \sqrt{\frac{3y-57}{2}} = \sqrt{\frac{7y-47}{2}}:$$

$\therefore y + \sqrt{6y^2 - 114y} + \dfrac{3y-57}{2} = \dfrac{7y-47}{2}$, by squaring both members;

$$\therefore \sqrt{6y^2 - 114y} = y + 5, \text{ and } 5y^2 - 124y = 25:$$

$$\therefore y^2 - \frac{124}{5}y + \left(\frac{62}{5}\right)^2 = 5 + \left(\frac{62}{5}\right)^2 = \frac{3969}{25};$$

$$\therefore y - \frac{62}{5} = \pm\frac{63}{5}, \text{ and } y = 25 \text{ and } -\frac{1}{5}:$$

$$\text{whence we get } x = \frac{y-7}{2} = 9 \text{ and } -\frac{18}{5};$$

the latter of which, being merely a root of solution, does not satisfy the equation.

164. By getting rid of factors common to both the members of equations as in article (151), the solutions of equations of higher dimensions may frequently be obtained by the preceding methods; but they will generally be subject to the imperfection alluded to in the said article.

Ex. 1. Given $x^3 - 3x = 2$, to find the values of $x$.

By adding $2x$ to both members, we have

$$x^3 - x = 2x + 2, \text{ or } x(x^2 - 1) = 2(x+1):$$

$\therefore$ by dividing both sides by $x+1$, we obtain

$$x(x-1) = 2, \text{ or } x^2 - x = 2:$$

$$\therefore x^2 - x + \frac{1}{4} = \frac{9}{4}, \text{ and } x - \frac{1}{2} = \pm\frac{3}{2};$$

$$\text{whence } x = 2 \text{ and } -1:$$

but in this example there is another root $= -1$, which has been lost sight of in consequence of the division by $x+1$, the proposed equation being equivalent to

$$(x+1)(x+1)(x-2) = 0,$$

which is satisfied twice by making $x = -1$.

Ex. 2. Given $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{x}{a}$, to find the values of $x$.

Here $ax+a\sqrt{x^2-a^2}=x^2-x\sqrt{x^2-a^2}$;

$$\therefore\ (a+x)\sqrt{x^2-a^2}=x^2-ax=x(x-a);$$

whence dividing both sides by $\sqrt{x-a}$, we obtain

$$(a+x)^{\frac{3}{2}}=x\sqrt{x-a},\ \text{and}\ \therefore\ (a+x)^3=x^3-ax^2;$$

$$\therefore\ 4ax^2+3a^2x=-a^3,\ \text{and}\ x^2+\frac{3a}{4}x=-\frac{a^2}{4};$$

$$\therefore\ x^2+\frac{3a}{4}x+\frac{9a^2}{64}=-\frac{7a^2}{64},\ \text{and}\ x+\frac{3a}{8}=\pm\frac{a}{8}\sqrt{-7},$$

from which we find $x=\dfrac{a}{8}\left\{\pm\sqrt{-7}-3\right\}$ : and, these values being imaginary, no possible solution has been obtained.

If however we resume the equation at the step where

$$(a+x)\sqrt{x^2-a^2}=a(x-a),$$

and put it in the form

$$(x+a)^{\frac{3}{2}}\sqrt{x-a}=x\sqrt{x-a}\sqrt{x-a},$$

$$\text{or}\ \left\{(x+a)^{\frac{3}{2}}-x\sqrt{x-a}\right\}\sqrt{x-a}=0,$$

it will obviously be satisfied either by making

$$(x+a)^{\frac{3}{2}}=x\sqrt{x-a}$$

as above, or by putting

$$\sqrt{x-a}=0\ \text{which gives}\ x=a;$$

and this last is a possible solution which has been entirely passed over by the step taken in the first part of the operation.

165. In what has preceded we have always supposed the equation to be reduced to the form

$$x^2 - px + q = 0,$$

where $p$ and $q$ may be positive or negative, integral or fractional quantities; but if in order to avoid fractions, we allow the highest power of $x$ to retain its coefficient, so that the equation is of the form

$$ax^2 + bx = c,$$

and multiply every term by four times the coefficient of the first term and add to both sides the square of that of the second, we shall have

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2,$$

the first member of which is obviously a complete square: hence, extracting the square roots of both sides, we get

$$2ax + b = \pm\sqrt{4ac + b^2}:$$

$$\therefore\ 2ax = -b \pm \sqrt{4ac + b^2},$$

$$\text{and } \therefore\ x = \frac{-b \pm \sqrt{4ac + b^2}}{2a};$$

and thus the values of $x$ are determined.

Since the solution above gives

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a} = \frac{-b \pm \left\{b + \dfrac{2ac}{b} - \dfrac{2a^2c^2}{b^3} + \&c.\right\}}{2a},$$

if we suppose $a = 0$, or the proposed equation to become $bx = c$, we find $x = \frac{c}{b}$ and $\infty$; the latter of which depends, for its existence, solely on the method of solution.

This is the *Hindoo* method of solving quadratics, and may be found in the *Beej Gunnit* or *Beja Ganita*, a treatise on Algebra written by *Bhasker Acharij* a famous mathematician who lived about the beginning of the thirteenth century.

Ex. Given $7x^2 - 13x = 2$, to find the values of $x$.

Multiplying all the terms by $4 \times 7$ or by 28, we have

$$196x^2 - 364x = 56;$$

$\therefore$ completing the square, $196x^2 - 364x + 169 = 225$:

whence by evolution, $14x - 13 = \pm 15$;

$$\therefore\ 14x = 28 \text{ and } -2;$$

$$\text{and } \therefore\ x = \frac{28}{14} = 2 \text{ and } \frac{1}{7};$$

which are the same as would have been found by the general method before given.

*Problems dependent upon Quadratic Equations.*

166. By proper attention to the observations and directions contained in (152), problems belonging to this head will be algebraically exhibited by means of equations of the second degree, or such as are reducible to that degree by substitutions, &c.; and it then only remains to apply the rules above laid down to obtain their solution.

As, however, every equation of the second order admits of two solutions at least, of which it frequently happens that only one will answer the conditions of the problem, it may be observed that in the process of reducing the equation, a new condition is sometimes introduced, and a corresponding new value of the unknown quantity, which did not originally belong to it; or that the algebraical expression of the equation is more comprehensive than the enunciation of the problem, so that it comprises other conditions besides those which are peculiar to the question under consideration; and consequently it will be necessary at last to reject such values of the unknown quantity as are excluded by the nature of the case.

This will be done in some of the following examples.

Ex. 1. Find two numbers, one of which is triple the other, and the sum of whose squares is 90.

Let $x=$ the less number; $\therefore\ 3x=$ the greater;

whence $x^2+9x^2=90$, or $10x^2=90$, by the question:

$$\therefore\ x^2=9,$$

and consequently $x=\pm 3$ and $3x=\pm 9$, the required numbers:

but if positive numbers be required, 3 and 9 will be those which fulfil the conditions.

Ex. 2. Divide the number 12 into two parts, so that the square of one of them may be four times as great as the square of the other.

Let $x=$ one of the parts; $\therefore\ 12-x=$ the other;

$$\therefore\ x^2=4(12-x)^2, \text{ by the question;}$$

$$\text{and } \therefore\ x=\pm 2(12-x)=\pm 24 \mp 2x,$$

$$\therefore\ x(1\pm 2)=\pm 24, \text{ and } x=\frac{\pm 24}{1\pm 2}:$$

whence, making use of the upper signs, we find that the parts are 8 and 4;

but if the negative signs were retained, the parts would be 24 and $-12$ which are both excluded by the nature of the case.

Ex. 3. Required a number which being multiplied by the square root of its half, shall give the result 54.

Assume $x$ to represent the number; then by the question, we shall have

$$x\times\sqrt{\frac{x}{2}}=54, \text{ or } x^{\frac{3}{2}}=54\sqrt{2}:$$

$\therefore\ x^3 = 5832$ and $x = 18$ the required number, which may easily be shewn to fulfil the condition.

Ex. 4. Find a number whose square exceeds it by 6.

Let $x$ represent the required number; then we shall have

$$x^2 - x = 6,$$

which is a quadratic equation exhibiting algebraically the proposed condition:

$$\therefore\ x^2 - x + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\text{whence } x - \frac{1}{2} = \pm\frac{5}{2}, \text{ and } \therefore\ x = 3 \text{ and } -2:$$

and both these values satisfy the condition of the equation, though, properly speaking, the second is excluded by the enunciation of the problem.

Ex. 5. Divide the number 20 into two parts so that the product of the whole and one of the parts may be equal to the square of the other part.

Let $2x =$ the difference of the two parts; then since $20 =$ their sum, we have by (34)

$$10 + x = \text{the greater part},$$

$$10 - x = \text{the less part}:$$

whence, by the question, is obtained the equation,

$$20(10 - x) = (10 + x)^2,$$

$$\text{or } 200 - 20x = 100 + 20x + x^2:$$

$$\therefore\ x^2 + 40x = 100,$$

$$\therefore\ x^2 + 40x + 400 = 500,$$

$$\text{and } \therefore\ x = -20 \pm \sqrt{500} = -20 \pm 10\sqrt{5}:$$

whereof the value $-20+10\sqrt{5}$ will alone be admissible, since the difference of the parts must obviously be a positive quantity:

$$\therefore \text{ the greater part } = 10 + x = 10\sqrt{5} - 10 = 10(\sqrt{5} - 1);$$

$$\text{and the less part } = 10 - x = 30 - 10\sqrt{5} = 10(3 - \sqrt{5}).$$

From these results it is evident that a number cannot be divided into two *rational* parts so as to answer the condition of the problem.

Ex. 6. Divide the quantity $a$ into two parts so that their product may be equal to $b^2$.

Let $x$ denote one of the parts; $\therefore a - x$ will represent the other:

$$\text{whence } x(a-x) = b^2, \text{ by the problem;}$$

$$\text{that is, } x^2 - ax = -b^2;$$

$$\therefore x^2 - ax + \frac{a^2}{4} = \frac{a^2 - 4b^2}{4}; \text{ and } x = \frac{a \pm \sqrt{a^2 - 4b^2}}{2},$$

$$\text{and } \therefore a - x = \frac{a \mp \sqrt{a^2 - 4b^2}}{2},$$

and these are the parts required, both of which answer the proposed condition.

Here it may be remarked that if $4b^2$ be greater than $a^2$, or $b$ be greater than $\frac{a}{2}$, the two parts become *unassignable* or *impossible:* and this circumstance shews, as hinted in (141), that the problem is impossible when the proposed product is greater than the square of half the given quantity; and also that this product is the greatest possible when

$$4b^2 = a^2 \text{ or } b = \frac{a}{2},$$

and therefore each of the required parts is half the quantity given.

Ex. 7. Required two numbers whose sum is 11, and the sum of whose squares is 61.

Let $x =$ one of the numbers; $\therefore\ 11 - x =$ the other:

$\therefore\ x^2 + (11 - x)^2 = 61$; that is, $2x^2 - 22x + 121 = 61$:

whence we have the following results,

$$x^2 - 11x = -30,\ x^2 - 11x + \frac{121}{4} = \frac{1}{4},\ \text{and}\ x - \frac{11}{2} = \pm\frac{1}{2};$$

consequently $x = 6$ and $5$,

and $\therefore\ 11 - x = 5$ and $6$:

and the required numbers are 6 and 5, the two roots of the equation which express the proposed condition.

Ex. 8. Divide $a$ into two parts so that the sum of the products arising from multiplying each by the square of the other may be equal to $b^3$.

Let $x =$ one of the parts; $\therefore\ a - x =$ the other part:

and the problem expressed algebraically gives the equation

$$x^2(a - x) + (a - x)^2 x = b^3:$$

whence $ax^2 - x^3 + a^2x - 2ax^2 + x^3 = b^3$, or $ax^2 - a^2x = -b^3$;

$$\therefore\ x^2 - ax + \frac{a^2}{4} = \frac{a^2}{4} - \frac{b^3}{a} = \frac{a^3 - 4b^3}{4a},$$

$$\text{and}\ x = \frac{a^2 \pm \sqrt{a^4 - 4ab^3}}{2a}:$$

wherefore the required parts are

$$\frac{a^2 \pm \sqrt{a^4 - 4ab^3}}{2a}\ \text{and}\ \frac{a^2 \mp \sqrt{a^4 - 4ab^3}}{2a};$$

and they will both be unassignable if $4ab^3$ be greater than $a^4$, or $b$ be greater than $\dfrac{a}{\sqrt[3]{4}}$.

Ex. 9. Given the sum of two magnitudes $= 2a$ and the sum of their cubes $= 2b^3$: to find them.

Let $2x$ represent the difference of the required magnitudes; then will the magnitudes themselves be $a + x$ and $a - x$: whence, by the question, we have

$$(a+x)^3 + (a-x)^3 = 2b^3:$$

$$\text{that is, } 2a^3 + 6ax^2 = 2b^3:$$

$$\therefore\ x^2 = \frac{b^3 - a^3}{3a} \text{ and } x = \pm\sqrt{\frac{b^3 - a^3}{3a}}:$$

and the magnitudes required will be

$$a + \sqrt{\frac{b^3 - a^3}{3a}} \text{ and } a - \sqrt{\frac{b^3 - a^3}{3a}},$$

each of the values of $x$ therefore equally answering the conditions proposed.

In order that these parts may be assignable, it is manifest that $b$ must not be less than $a$; and the extreme case in which the problem will be possible is when $b = a$, and therefore when the magnitudes are equal to each other.

Ex. 10. A farmer purchased a number of oxen for £112. and observed that if he had had one more for the same money, each of them would have cost him £2. less: required the number he purchased and the price of each.

Let $x =$ the number of oxen purchased;

$$\text{then will } \frac{112}{x} = \text{the price of each}:$$

$$\therefore\ (x+1) \times \left(\frac{112}{x} - 2\right) = 112, \text{ by the question}:$$

$$\text{that is, } x^2 + x = 56:$$

$$\therefore\ x^2 + x + \tfrac{1}{4} = 56 + \tfrac{1}{4} = \frac{225}{4}:$$

$$\text{whence } x + \tfrac{1}{2} = \pm\frac{15}{2}, \text{ and } x = 7 \text{ and } 8,$$

and it will be found upon trial that the former alone will answer the condition of the problem: that is, the number of oxen is 7, and therefore the price of each $= \dfrac{112}{7} =$ £16.

In this instance the second value of $x$, though it does not satisfy the proposed condition, is nevertheless not without a significant meaning, as it obviously solves the following problem:

A farmer purchased a number of oxen for £112., and observed that if he had had one *fewer* for the same money, each of them would have cost him £2. *more:* required the number he purchased and the price of each:

so that the algebraical expression above found, comprehending both these problems, is more general than the enunciation of either.

Ex. 11. Of a number of bees, after eight ninths and the square root of half of them had flown away, there were two remaining: what was the number at first?

Let $2x^2$ represent the number at first; then by the question, we have

$$\frac{16x^2}{9} + x + 2 = 2x^2:$$

$$\text{that is, } 2x^2 - 9x = 18:$$

$$\therefore\ 16x^2 - 72x + 81 = 144 + 81 = 225:$$

$$\text{whence } 4x - 9 = \pm 15,\ 4x = 24 \text{ and } -6,$$

$$\text{and } \therefore\ x = 6 \text{ and } -\frac{3}{2}:$$

wherefore, since the former value alone answers the conditions proposed, we have the required number $= 2.36 = 72$.

Ex. 12. *A* and *B* with dispatches between two towns start at the same time and meet when *A* has travelled 30 miles more than *B*, and they afterwards reach their destinations in 4 and 9 hours respectively: find the distance between the two towns and the rates of travelling of *A* and *B*.

Let $x$ = distance travelled by *B* before meeting;

$\therefore$ $x+30$ = distance travelled by *A*, and $2x+30$ = distance between the towns:

$\therefore$ *A* and *B* travel $x$ and $x+30$ miles in 4 and 9 hours respectively: whence their rates of travelling per hour are $\frac{x}{4}$ and $\frac{x+30}{9}$ respectively:

$\therefore$ *A*'s time between the two towns $= (2x+30) \div \frac{x}{4} = \frac{8x+120}{x}$,

$$\text{and } B\text{'s time} = (2x+30) \div \left(\frac{x+30}{9}\right) = \frac{18x+270}{x+30}:$$

wherefore since *B* takes 5 hours more than *A* to perform the journey, we must have

$$\frac{18x+270}{x+30} = \frac{8x+120}{x} + 5,$$

which is the algebraical expression of the problem: hence by reduction, we get

$$x^2 - 48x = 720, \text{ and } \therefore x = 60 \text{ and } -12,$$

of which the latter is manifestly excluded:

$\therefore$ the distance between the two towns $= 2x+30 = 150$ miles; and the rates of travelling of *A* and *B* represented by $\frac{x}{4}$ and $\frac{x+30}{9}$ are 15 and 10 miles an hour respectively.

Ex. 13. Three persons $A$, $B$ and $C$ can perform a piece of work in a certain time: $A$ alone could do it in $a$ hours more, $B$ alone in $b$ hours more, and $C$ alone in twice the time: how long would it occupy them jointly and singly?

Let $x$ be the number of hours they are occupied jointly;

$\therefore$ $x+a$, $x+b$ and $2x$ are the respective numbers of hours in which $A$, $B$ and $C$ can perform the work singly: hence, if unity be taken to represent the work,

$$\frac{1}{x+a},\ \frac{1}{x+b} \text{ and } \frac{1}{2x}$$

will be the parts of it performed by $A$, $B$ and $C$ in one hour, and

$$\therefore\ \frac{x}{x+a},\ \frac{x}{x+b} \text{ and } \frac{1}{2}$$

are the portions done by them in $x$ hours: hence, by the question, we shall have

$$\frac{x}{x+a}+\frac{x}{x+b}+\frac{1}{2}=1:$$

from which by reduction, &c. we find

$$x=\pm\sqrt{\left(\frac{a+b}{6}\right)^2+\frac{ab}{3}}-\frac{a+b}{6},$$

and the nature of the problem excludes the negative value: hence the times required are

$$\sqrt{\left(\frac{a+b}{6}\right)^2+\frac{ab}{3}}-\frac{a+b}{6};$$

$$\sqrt{\left(\frac{a+b}{6}\right)^2+\frac{ab}{3}}+\frac{5a-b}{6},\quad \sqrt{\left(\frac{a+b}{6}\right)^2+\frac{ab}{3}}+\frac{5b-a}{6},$$

$$\text{and } 2\sqrt{\left(\frac{a+b}{6}\right)^2+\frac{ab}{3}}-\frac{a+b}{3}.$$

Ex. 14. It is required to decompose the quantity $a$ into two factors, whose sum shall be the least possible.

Let the factors be $x$ and $\frac{a}{x}$, and assume $x+\frac{a}{x}=m$:

$$\therefore\ x^2-mx+\frac{m^2}{4}=\frac{m^2-4a}{4},$$

$$\text{and } x=\frac{m\pm\sqrt{m^2-4a}}{2}:$$

now, in order that the factors $x$ and $\frac{a}{x}$ may be possible, it is obvious that $m^2$ cannot be less than $4a$; and for the extreme case in which they are possible, we obtain

$$x=\frac{m}{2}=\sqrt{a},$$

so that the required factors are each equal to $\sqrt{a}$.

This result may also be obtained from the following consideration.

Let one of the factors $x=c\sqrt{a}$, and $\therefore$ the other $\frac{a}{x}=\frac{\sqrt{a}}{c}$;

whence we have $x+\frac{a}{x}=c\sqrt{a}+\frac{\sqrt{a}}{c}=\left(c+\frac{1}{c}\right)\sqrt{a}$,

which will manifestly be greater than $2\sqrt{a}$, unless $c=1$, as appears from (94).

Ex. 15. Given two magnitudes $a$ and $b$, whereof $a$ is the greater, to find the greatest possible value of which the expression $\frac{(x+a)(x-b)}{x^2}$ admits.

Let $\frac{(x+a)(x-b)}{x^2} = m$, from which there is immediately deduced by the ordinary process

$$x = \frac{a-b \pm \sqrt{(a+b)^2 - 4mab}}{2(m-1)}:$$

therefore, in order that the values of $x$ may be real, we must have $(a+b)^2$ equal to or greater than $4mab$, and $\therefore$ $m$ equal to or less than $\frac{(a+b)^2}{4ab}$:

in the extreme case $m = \frac{(a+b)^2}{4ab}$, which gives $m-1 = \frac{(a-b)^2}{4ab}$:

$$\text{whence } x = \frac{a-b}{2} \times \frac{4ab}{(a-b)^2} = \frac{2ab}{a-b}:$$

and the corresponding value which the proposed expression admits of will obviously be $\frac{(a+b)^2}{4ab}$.

Also, if for $x$ we put $\frac{2ab}{a-b} + \delta$, we shall have

$$\frac{(x+a)(x-b)}{x^2} = \frac{(a+b)^2}{4ab} - \frac{(a-b)^2\delta^2}{4ab\left(\frac{2ab}{a-b} + \delta\right)},$$

which shews that $\frac{(a+b)^2}{4ab}$ is the greatest value required.

## III. ELIMINATION OF UNKNOWN QUANTITIES.

167. From what has been already said, it is evident that the value of any one of the symbols concerned in an equation is entirely dependent upon those of the rest, and it will become *known* only when the values of the rest are *given* or *assigned*.

If then, there exist *simultaneously* two or more equations established among the same symbols, it is obvious that the determination of any one of them cannot be effected without reference to the assignability of the rest; and the operations before explained will in general enable us to effect, by means of the relations and connections subsisting among them, such modifications and changes that one or more of the unknown quantities shall not be found in the equations which are made to result. Quantities thus treated are said to be *eliminated* or *exterminated*, and the resulting equations will manifestly involve fewer unknown magnitudes than are found in the equations proposed. Hence, if the numbers of equations and unknown quantities be properly adjusted, it is obvious that the given equations may all be reduced to one final equation, in which there is found only one unknown magnitude combined with coefficients that are supposed known.

This part of Algebra being of a very intricate nature, and the requisite operations in general exceedingly embarrassing, it will here be entered upon only so far as is necessary to the prosecution of the subjects contained in the subsequent articles of the present chapter.

168. *First Method.* Any number of equations of this description being proposed, from which it is required to eliminate or exterminate one or more unknown quantities, the most natural and obvious mode of proceeding would be to endeavour by means of one of them to express the value of the unknown quantity least involved, in terms of the rest and known coefficients: and by its substitution in the equations remaining, it manifestly follows that one of the unknown quantities will not be found in the equations which thence arise: so that if the number of equations be sufficient, a repetition of the same operation will clearly lead to an equation independent of all the unknown quantities with the exception of one.

This method will manifestly be very limited in its operations, in consequence of no general solution of equations of

higher orders than the second having as yet been investigated; but it shall be illustrated by such examples as are of most frequent occurrence in practice.

Ex. 1. Given the two equations

$$ax + by = c \text{ and } dx + ey = f:$$

then if it be required to eliminate $x$,

we have from the former $ax = c - by$,

$$\text{and } \therefore\ x = \frac{c - by}{a}:$$

wherefore by the substitution of this value of $x$ in the latter, we get

$$d\left(\frac{c - by}{a}\right) + ey = f,$$

$$\text{or } (ae - bd)y = af - cd,$$

an equation involving only the unknown quantity $y$, the other unknown quantity $x$ having been thus eliminated.

A similar process would manifestly have eliminated $y$.

Ex. 2. Given the simultaneous equations $x^2 + y^2 = a^2$ and $x^2 + bxy + y^2 = 0$, to eliminate $x$.

From the former of the proposed equations we have

$$x^2 = a^2 - y^2, \text{ and } \therefore\ x = \pm\sqrt{a^2 - y^2}:$$

whence, by substitution in the latter, we get

$$a^2 - y^2 \pm by\sqrt{a^2 - y^2} + y^2 = 0,$$

$$\text{or } a^2 \pm by\sqrt{a^2 - y^2} = 0;$$

$\therefore$ by transposing and squaring both sides, we obtain

$$b^2y^4 - a^2b^2y^2 + a^4 = 0,$$

which is the final equation involving only $y$.

Also, since $x$ and $y$ are involved in precisely the same manner, the final equation in terms of $x$ would be

$$b^2x^4 - a^2b^2x^2 + a^4 = 0.$$

Ex. 3. Let there be proposed the three following equations;

$$ax + by + cz = d,$$

$$ex + fy + gz = h,$$

$$kx + ly + mz = n:$$

then from the first of these we have

$$x = \frac{d - by - cz}{a};$$

whence, by substitution in the second and third, we obtain

$$e\left(\frac{d - by - cz}{a}\right) + fy + gz = h,$$

$$\text{and } k\left(\frac{d - by - cz}{a}\right) + ly + mz = n;$$

wherefore, by clearing of fractions, there immediately result

$$(af - be)y + (ag - ce)z = ah - de,$$

$$\text{and } (al - bk)y + (am - ck)z = an - dk:$$

thus have we eliminated the unknown quantity $x$, and the preceding examples shew us how, by means of these two equations, either $y$ or $z$ may be then exterminated, and a final equation involving only one of them be obtained. Similarly of more equations.

169. *Second Method.* It is obvious, that the elimination of an unknown quantity may be effected by instituting an equality between its values determined from different equations, since its exclusion from both members of the resulting equations will necessarily ensure its extermination altogether.

This method will be illustrated by the following examples, but possesses the same defect as the last.

Ex. 1. Let $ax+by=c$ and $dx+ey=f$, as before: then, from these equations, we obtain

$$x=\frac{c-by}{a} \text{ and } x=\frac{f-ey}{d}:$$

whence, in consequence of the simultaneous existence of the equations, we shall manifestly have

$$\frac{c-by}{a}=\frac{f-ey}{d},$$

$$\text{and } \therefore \ cd-bdy=af-aey,$$

$$\text{or } (ae-bd)y=af-cd, \text{ which does not involve } x.$$

Ex. 2. Given the two equations $x^2-axy=by$ and $y^2-xy=c^2$, to eliminate $x$.

From the first, $x^2-ayx+\dfrac{a^2y^2}{4}=\dfrac{a^2y^2+4by}{4}$,

$$\text{and } \therefore \ x=\frac{ay\pm\sqrt{a^2y^2+4by}}{2},$$

$$\text{also, from the second, } x=\frac{y^2-c^2}{y}:$$

$$\therefore \text{ we must have } \frac{ay\pm\sqrt{a^2y^2+4by}}{2}=\frac{y^2-c^2}{y}:$$

whence, by reduction, is easily deduced

$$(a-1)y^4+by^3-(a-2)c^2y^2-c^4=0,$$

an equation involving only $y$ and known quantities.

Ex. 3. From the three simultaneous equations,

$$ax + by + cz = d,$$
$$ex + fy + gz = h,$$
$$kx + ly + mz = n,$$

we immediately obtain the following expressions for $x$, viz.;

$$x = \frac{d - by - cz}{a},$$

$$x = \frac{h - fy - gz}{e},$$

$$x = \frac{n - ly - mz}{k}:$$

wherefore this method gives us the two equations

$$\frac{d - by - cz}{a} = \frac{h - fy - gz}{e},$$

$$\text{and } \frac{d - by - cz}{a} = \frac{n - ly - mz}{k};$$

which, by reduction, lead respectively to the equations,

$$(af - be)y + (ag - ce)z = ah - de,$$
$$\text{and } (al - bk)y + (am - ck)z = an - dk;$$

so that $x$ has been eliminated: and, by pursuing the steps of the last examples, we may get rid of either $y$ or $z$, and thus obtain an equation involving only one of the symbols $z$ and $y$.

170. *Third Method.* Another method of elimination, not essentially different from the two preceding, still remains to be explained: and this is by multiplying or dividing the members of the equations proposed by such quantities as will render the coefficients of one or more of the unknown quantities in two or more of them equal to one another, and then taking their sum or difference as the case may require.

The following examples of this method will be sufficient for a full explanation of it.

Ex. 1. Given the equations $ax+by=d$ and $ex-fy=g$; then if the members of the first equation be multiplied by $f$, and those of the second by $b$, we have

$$afx+bfy=df,$$

$$\text{and } bex-bfy=bg;$$

$\therefore$ by addition, $(af+be)x=bg+df$, so that $y$ is eliminated:

again, multiplying the terms of the first equation by $e$ and those of the second by $a$, we get

$$aex+bey=de,$$

$$\text{and } aex-afy=ag;$$

$\therefore$ by subtraction, $(af+be)y=de-ag$, which does not involve $x$.

Ex. 2. Given the two equations $\frac{x-a}{x}=\frac{(y-b)^2}{y^2}$ and $\frac{x-c}{x}=\frac{(y-d)^2}{y^2}$, to eliminate $x$ and $y$.

From the first, $1-\frac{a}{x}=1-\frac{2b}{y}+\frac{b^2}{y^2}$,

$$\text{and } \therefore \frac{a}{x}=\frac{2b}{y}-\frac{b^2}{y^2}:$$

$$\text{from the second, } 1-\frac{c}{x}=1-\frac{2d}{y}+\frac{d^2}{y^2},$$

$$\text{and } \therefore \frac{c}{x}=\frac{2d}{y}-\frac{d^2}{y^2}:$$

multiplying the first of these by $d$ and the second by $b$, we have

$$\frac{ad}{x}=\frac{2bd}{y}-\frac{b^2d}{y^2},$$

$$\frac{bc}{x}=\frac{2bd}{y}-\frac{d^2b}{y^2}:$$

$\therefore$ by subtraction, is obtained

$$\frac{ad-bc}{x}=\frac{bd^2-b^2d}{y^2}=\frac{bd(d-b)}{y^2},$$

or $\dfrac{x}{ad-bc}=\dfrac{y^2}{bd(d-b)}$, by inverting both sides:

again, multiplying the first by $d^2$ and the second by $b^2$, we get

$$\frac{ad^2}{x}=\frac{2bd^2}{y}-\frac{b^2d^2}{y^2},$$

$$\frac{b^2c}{x}=\frac{2b^2d}{y}-\frac{b^2d^2}{y^2}:$$

whence, by subtraction, as before, we have

$$\frac{ad^2-b^2c}{x}=\frac{2(bd^2-b^2d)}{y}=\frac{2bd(d-b)}{y}:$$

wherefore, multiplying together the corresponding members of these two resulting equations, we arrive at

$$\frac{x}{ad-bc}\times\frac{ad^2-b^2c}{x}=\frac{y^2}{bd(d-b)}\times\frac{2bd(d-b)}{y},$$

$$\text{or } \frac{ad^2-b^2c}{ad-bc}=2y,$$

which involves only the unknown magnitude $y$:

also, $\dfrac{ad^2-b^2c}{x}\times\dfrac{ad^2-b^2c}{ad-bc}=\dfrac{2bd(d-b)}{y}\times 2y=4bd(d-b)$,

or $\dfrac{ad-bc}{(ad^2-b^2c)^2}x=\dfrac{1}{4bd(d-b)}$, which contains only $x$.

Ex. 3. Given the three simultaneously existent equations:

$$x + y + z = a,$$

$$xy + xz + yz = b^2,$$

$$xyz = c^3,$$

to eliminate any two of the symbols as $y$ and $z$.

Multiplying the first by $x^2$, we get

$$x^3 + x^2 y + x^2 z = a x^2:$$

and multiplying the second by $x$, we obtain

$$x^2 y + x^2 z + xyz = b^2 x:$$

whence, taking the latter of these results from the former, we have

$$x^3 - xyz = ax^2 - b^2 x,$$

to which if we add the third, there arises

$$x^3 = ax^2 - b^2 x + c^3,$$

and $\therefore$ $x^3 - ax^2 + b^2 x - c^3 = 0$ is the final equation involving only the unknown quantity $x$.

171. The three methods of elimination above explained and exemplified, are those most commonly resorted to in practice, and by judicious combinations of them, many other modes will suggest themselves in the consideration of particular instances.

To eliminate one and two unknown quantities, it has appeared in the preceding examples, that two and three independent equations respectively are *necessary* and *sufficient;* and it evidently follows from an extension of the views which have been taken of particular cases, that by means of $n$ equations independent of one another, any $n-1$ of the unknown quantities involved in them may be made to disappear from the equation which finally results.

If one or more of the equations be derivable from the others, these will obviously answer no purpose which is not equally answered without them; and thus, in general, $n$ independent equations will be requisite at the same time that they are sufficient for the extermination of $n-1$ unknown quantities.

The general theory of elimination being of far too difficult a nature to admit of its introduction into an elementary treatise like the present, for further information upon the subject the reader is referred to the article *Elimination* in the second volume of BONNYCASTLE's Algebra, to the Algebra of EULER, to the *Theorie generale des Equations Algebriques* of BEZOUT and to the AUTHOR's *Theory of Equations.*

*Equations involving two or more unknown Quantities.*

172. It has been shewn in some of the preceding articles, that, by means of a proper number of equations, all the unknown quantities except one involved in them may be eliminated; and it is further manifest that the final equation involving that one may always be reduced to one of the forms specified in (148): and when the equation thus resulting is of the first or second order, or capable of reduction to either of those orders by substitutions or otherwise, the application of the methods already investigated and exemplified will lead immediately to its solution: and one or more of the unknown quantities being thus discovered, the rest may be determined by substitution in the expressions representing their values.

From the want of a general solution of equations containing only one unknown magnitude, the solution of equations containing more than one must necessarily be very limited, and it will be sufficiently illustrated by the following examples.

Ex. 1. Given $7x+10y=41$ and $13x-11y=17$, to find the values of $x$ and $y$.

From the first equation we find $x = \frac{41-10y}{7}$; whence, by the substitution of this value for $x$ in the second equation, according to (168), we obtain

$$13\left(\frac{41-10y}{7}\right) - 11y = 17,$$

$$\text{and } \therefore\ 533 - 207y = 119, \text{ by reduction:}$$

whence we have

$$207y = 414 \text{ and } \therefore\ y = \frac{414}{207} = 2:$$

$$\text{wherefore } x = \frac{41-10y}{7} = \frac{41-20}{7} = \frac{21}{7} = 3:$$

and the values 3 and 2 of $x$ and $y$ will be found to satisfy simultaneously both the proposed equations.

Ex. 2. Given

$$\frac{x+2}{7} + \frac{y-x}{4} = 2x - 8 \text{ and } \frac{2y-3x}{3} + 2y = 3x + 4,$$

to find the values of $x$ and $y$.

By reduction, the proposed equations become

$$59x - 7y = 232 \text{ and } 3x - 2y = -3:$$

$$\therefore \text{ the former gives } y = \frac{59x - 232}{7},$$

$$\text{and the latter gives } y = \frac{3x+3}{2};$$

whence, according to (169), we have

$$\frac{59x - 232}{7} = \frac{3x+3}{2},$$

$$\therefore\ 97x = 485 \text{ and } x = \frac{485}{97} = 5:$$

also, from the latter reduced equation, which is the simpler of the two, we find

$$y = \frac{3x+3}{2} = \frac{15+3}{2} = \frac{18}{2} = 9:$$

and by trial we may easily be assured that 5 and 9 answer the conditions.

Ex. 3. Given the two simultaneous equations:

$$8x - 3y = 19 + \frac{7x+1}{8}$$

$$\text{and } 3x - 5y = 18 - \frac{13y-3}{11},$$

to find the values of $x$ and $y$.

Clearing the equations of fractions, &c. we have

$$19x - 8y = 51 \text{ and } 11x - 14y = 67:$$

$\therefore$ multiplying these equations by 11 and 19 respectively, according to (170), we get from the first, $209x - 88y = 561$,

and from the second, $209x - 266y = 1273$;

$\therefore$ by subtracting the latter of these from the former, we find

$$178y = -712 \text{ and } \therefore\ y = -\frac{712}{178} = -4:$$

$$\text{whence } 19x = 51 + 8y = 51 - 32 = 19 \text{ and } \therefore\ x = \frac{19}{19} = 1:$$

and the required values of $x$ and $y$ are therefore 1 and $-4$.

Ex. 4. Given $x^2 + xy = 66$ and $x^2 - y^2 = 11$, to find the values of $x$ and $y$.

From the first equation, $y^2 = \left(\frac{66 - x^2}{x}\right)^2$,

$\therefore$ by substitution in the second, $x^2 - \left(\frac{66 - x^2}{x}\right)^2 = 11$;

$$\therefore\ 4356 - 132x^2 + x^4 = x^4 - 11x^2:$$

$$\text{whence } 121x^2 = 4356,$$

$$\therefore\ x^2 = 36 \text{ and } x = \pm 6:$$

$$\text{therefore } y^2 = x^2 - 11 = 36 - 11 = 25,$$

$$\text{and } y = \pm 5.$$

Ex. 5. Given $x^2 + y^2 = 34$ and $x^2 - xy = 10$, to find the values of $x$ and $y$.

From the first equation, $y^2 = 34 - x^2$,

and from the second, $y^2 = \left(\frac{x^2 - 10}{x}\right)^2$;

$$\therefore\ 34 - x^2 = \frac{x^4 - 20x^2 + 100}{x^2},$$

$$\text{and } x^4 - 27x^2 = -50:$$

$$\text{whence } x^4 - 27x^2 + \frac{729}{4} = \frac{529}{4}, \text{ and } \therefore\ x^2 - \frac{27}{2} = \pm \frac{23}{2};$$

$\therefore$ $x^2 = 25$ and $2$, and consequently $x = \pm 5$ and $\pm \sqrt{2}$:

$$\text{wherefore } y = \frac{x^2 - 10}{x} = \pm 3 \text{ and } \pm 4\sqrt{2}.$$

Ex. 6. Given $x + 2y + 3z = 14$, $2x - 3y + 4z = 8$ and $3x + 4y - 5z = -4$, to find the values of $x$, $y$ and $z$.

From the first equation, $x = 14 - 2y - 3z$,

from the second, $x = \frac{8 + 3y - 4z}{2}$,

and from the third, $x = \dfrac{-4-4y+5z}{3}$:

whence we shall manifestly have the two equations

$$14-2y-3z = \frac{8+3y-4z}{2}$$

$$\text{and } 14-2y-3z = \frac{-4-4y+5z}{3};$$

and these involve only two unknown quantities $y$ and $z$: again, these equations by reduction become

$$7y+2z=20$$

$$\text{and } y+7z=23:$$

hence, multiplying the latter by 7, we obtain

$$7y+49z=161,$$

also, from the former, $7y+\ \ 2z=20$;

$\therefore$ by subtraction, we get $47z=141$ and $\therefore$ $z=\dfrac{141}{47}=3$:

$$\therefore\ y=23-7z=23-21=2,$$

$$\text{and } x=14-2y-3z=14-4-9=1;$$

and the numbers 1, 2 and 3 will be found simultaneously to satisfy the three equations proposed.

Ex. 7. Given

$$x^2+y^2+z^2=a^2,\ y^2-2xz=b^2 \text{ and } cx+dz=e^2,$$

to find the values of $x$, $y$ and $z$.

From the last equation, $x=\dfrac{e^2-dz}{c}$, $\therefore$ $x^2=\left(\dfrac{e^2-dz}{c}\right)^2$:

and from the second, $y^2=b^2+2xz=b^2+\dfrac{2e^2z-2dz^2}{c}$:

$\therefore$ by the substitution of these values of $x^2$ and $y^2$ in the first equation, we get

$$\frac{(e^2-dz)^2}{c^2}+b^2+\frac{2e^2z-2dz^2}{c}+z^2=a^2:$$

whence by reduction

$$(c-d)^2z^2+2e^2(c-d)z=(a^2-b^2)c^2-e^4,$$

$$\text{and } \therefore\ z^2+\frac{2e^2z}{c-d}=\frac{(a^2-b^2)c^2-e^4}{(c-d)^2}:$$

$$\therefore\ z^2+\frac{2e^2z}{c-d}+\frac{e^4}{(c-d)^2}=\frac{(a^2-b^2)c^2}{(c-d)^2},$$

$$\text{and } z+\frac{e^2}{c-d}=\pm\frac{c\sqrt{a^2-b^2}}{c-d};$$

$$\text{whence } z=\frac{-e^2\pm c\sqrt{a^2-b^2}}{c-d}:$$

$$\therefore\ x=\frac{e^2-dz}{c}=\frac{e^2}{c}-\frac{d}{c}\left\{\frac{-e^2\pm c\sqrt{a^2-b^2}}{c-d}\right\}$$

$$=\frac{ce^2-de^2+de^2\mp cd\sqrt{a^2-b^2}}{c(c-d)}$$

$$=\frac{e^2\mp d\sqrt{a^2-b^2}}{c-d}:$$

$$\therefore\ y^2=b^2+2xz$$

$$=b^2+2\left\{\frac{(e^2\mp d\sqrt{a^2-b^2})(-e^2\pm c\sqrt{a^2-b^2})}{(c-d)^2}\right\}$$

$$=\frac{b^2(c+d)^2-2(a^2+b^2)cd-2e^2\{e^2\mp(c+d)\sqrt{a^2-b^2}\}}{(c-d)^2},$$

and thence

$$y = \frac{\pm\sqrt{b^2(c+d)^2 - 2(a^2+b^2)cd - 2e^2\{e^2 \mp (c+d)\sqrt{a^2-b^2}\}}}{c-d}.$$

Ex. 8. Given $ax + by = a^2$, $bx - az = b^2$, $cx + du = c^2$ and $dy - cz = d^2$, to find the values of $x$, $y$, $z$ and $u$.

From the first equation, $abx + b^2y = a^2b$,

and from the second, $abx - a^2z = ab^2$;

$\therefore$ by subtraction, $b^2y + a^2z = ab(a-b)$:

and $\therefore$ by multiplication, $b^2dy + a^2dz = abd(a-b)$,

also, from the fourth, $b^2dy - b^2cz = b^2d^2$;

$\therefore$ by subtraction, $(a^2d + b^2c)z = bd(a^2 - ab - bd)$,

$$\text{and thence } z = \frac{bd(a^2-ab-bd)}{a^2d+b^2c}:$$

again, from the second equation,

$$bx = b^2 + az = b^2 + \frac{abd(a^2-ab-bd)}{a^2d+b^2c}$$

$$= \frac{b(a^3d + b^3c - abd^2)}{a^2d+b^2c},$$

$$\text{and thence } x = \frac{a^3d + b^3c - abd^2}{a^2d+b^2c}:$$

$$\text{also, from the fourth, } dy = d^2 + cz = d^2 + \frac{bcd(a^2-ab-bd)}{a^2d+b^2c}$$

$$= \frac{d(a^2d^2 + a^2bc - ab^2c)}{a^2d+b^2c},$$

$$\text{and thence } y = \frac{a^2d^2 + a^2bc - ab^2c}{a^2d+b^2c}:$$

and from the third, $du = c^2 - cx = c^2 - \dfrac{c(a^3d + b^3c - abd^2)}{a^2d + b^2c}$

$$= \frac{acd(ac + bd - a^2) - b^2c^2(b - c)}{a^2d + b^2c},$$

$$\text{and } \therefore\ u = \frac{acd(ac + bd - a^2) - b^2c^2(b - c)}{(a^2d + b^2c)d}:$$

and thus the four unknown quantities are determined.

And similarly of more unknown quantities.

173. In the great variety of equations that are met with involving two or more unknown quantities, the ordinary processes of elimination may frequently be dispensed with, and recourse be had to artifices which will in most cases greatly abridge the labour of solution. The discovery of such artifices must, however, be left to the student's ingenuity, though it may be observed that they usually depend upon the fundamental rules of addition, subtraction, &c. &c.

The following examples will exhibit some of them.

Ex. 1. Given $x^2 + y^2 = 52$ and $xy = 24$, to find the values of $x$ and $y$.

From the first, $x^2 + y^2 = 52$;

from the second, $2xy = 48$;

$\therefore$ by addition, $x^2 + 2xy + y^2 = 100$,

and by subtraction, $x^2 - 2xy + y^2 = 4$:

$\therefore$ by evolution we get $x + y = \pm 10$,

and $x - y = \pm 2$:

whence, by addition and subtraction again, we obtain

$$2x = \pm 12,\ 2y = \pm 8,$$

$$\text{and } \therefore\ x = \pm 6 \text{ and } y = \pm 4.$$

Ex. 2. Given $x^2 - xy = a^2$ and $xy - y^2 = b^2$, to find the values of $x$ and $y$.

From the 1st, $x(x-y) = a^2$;

from the 2nd, $y(x-y) = b^2$;

whence, by division, we obtain $\frac{x}{y} = \frac{a^2}{b^2}$, or $y = \frac{b^2}{a^2}x$:

$\therefore$ by substitution, we have $x^2 - \frac{b^2}{a^2}x^2 = a^2$,

or $(a^2 - b^2)x^2 = a^4$;

whence $x = \pm\frac{a^2}{\sqrt{a^2 - b^2}}$,

and $\therefore$ $y = \frac{b^2}{a^2}x = \pm\frac{b^2}{\sqrt{a^2 - b^2}}$.

Ex. 3. Given $\frac{x}{y} - \frac{y}{x} = \frac{x+y}{x^2+y^2}$ and $\frac{x^2}{y^2} - \frac{y^2}{x^2} = \frac{x-y}{y^2}$, to find the values of $x$ and $y$.

From the first equation, $x^4 - y^4 = xy(x+y)$,

and from the second, $x^4 - y^4 = x^2(x-y)$:

$\therefore$ we shall have $xy(x+y) = x^2(x-y)$;

whence $y(x+y) = x(x-y)$,

and $\therefore$ $2xy = x^2 - y^2$:

now $(x^2 - y^2)(x^2 + y^2) = xy(x+y)$,

$\therefore$ $2xy(x^2 + y^2) = xy(x+y)$,

whence $x^2 + y^2 = \frac{x+y}{2}$:

$$\text{but } (x^2-y^2)(x^2+y^2)=x^2(x-y);$$

$$\therefore\ (x^2-y^2)\left(\frac{x+y}{2}\right)=x^2(x-y),$$

$$\text{whence } (x+y)^2=2x^2,$$

$$\text{and } \therefore\ x+y=\pm x\sqrt{2}:$$

here, for the sake of conciseness, we shall use only the positive sign, and then we have

$$y=(\sqrt{2}-1)x, \text{ and thence } y^2=(3-2\sqrt{2})x^2:$$

$$\text{therefore, since } x^2+y^2=\frac{x+y}{2},$$

$$\text{we have } (4-2\sqrt{2})x^2=\frac{x}{\sqrt{2}},$$

$$\text{from which we get } x=\frac{1}{\sqrt{2}(4-2\sqrt{2})}=\frac{1}{4(\sqrt{2}-1)}:$$

$$\text{and thence } y=(\sqrt{2}-1)x=\frac{\sqrt{2}-1}{4(\sqrt{2}-1)}=\frac{1}{4}.$$

Ex. 4. Given

$$x^4+y^4=1+2xy+3x^2y^2 \text{ and } x^3+y^3=2xy^2+2y^2+x+1,$$

to find the values of $x$ and $y$.

$$\text{From the first we have } x^4-2x^2y^2+y^4=1+2xy+x^2y^2,$$

$$\text{and } \therefore \text{ by evolution, } x^2-y^2=1+xy:$$

$$\text{from the second we have } x^3-2xy^2+y^3=2y^2+x+1,$$

$$\text{that is, } (x-y)(x^2+xy-y^2)=2y^2+x+1:$$

$\therefore$ substituting for $x^2-y^2$ its value $1+xy$, we get

$$(x-y)(1+2xy)=2y^2+x+1,$$

$$\text{and } \therefore\ 2y\,(x^2 - xy - y) = y + 1:$$

$$\text{but since } x^2 - xy = y^2 + 1,$$

$$\therefore\ 2y\,(y^2 - y + 1) = y + 1,$$

$$\text{and } \therefore\ y\,(1 + 2y^2) = 1 + 2y^2, \text{ whence } y = 1;$$

$$\text{also, since } x^2 - y^2 = 1 + xy, \text{ we have } x^2 - 1 = x + 1,$$

$$\text{and } \therefore\ x - 1 = 1 \text{ or } x = 2.$$

Other values might have been found by retaining the double sign, in addition to which many quantities which would satisfy the equations have been passed over by the several equal divisions of the members.

Ex. 5. Given $\dfrac{x - \sqrt{x^2 - y^2}}{x + \sqrt{x^2 - y^2}} = x$ and $\dfrac{x}{y} = \sqrt{\dfrac{1 + x}{1 + y}}$, to find the values of $x$ and $y$.

From the first equation, $x\,(1 - x) = (1 + x)\sqrt{x^2 - y^2}$:

$$\text{and from the second, } x^2 - y^2 = y^2 x - x^2 y,$$

$$\therefore\ x + y = -xy \text{ and } y = -\frac{x}{1 + x};$$

$$\therefore\ x^2 - y^2 = x^2 - \frac{x^2}{(1 + x)^2} = \frac{x^2}{(1 + x)^2}(2x + x^2);$$

whence, by substitution, we get

$$x\,(1 - x) = (1 + x)\frac{x}{1 + x}\sqrt{2x + x^2} = x\sqrt{2x + x^2};$$

$$\therefore\ 1 - x = \sqrt{2x + x^2},$$

$$\text{and } 1 - 2x + x^2 = 2x + x^2;$$

$$\therefore\ x = \frac{1}{4}:$$

$$\text{wherefore } y = -\frac{x}{1+x} = -\frac{1}{5}:$$

and to this solution the same remarks may be applied as to the last.

Ex. 6. Given $x + \sqrt{3y^2 - 11 + 2x} = 7 + 2y - y^2$ and $\sqrt{3y - x + 7} = \frac{x+y}{x-y}$, to find the values of $x$ and $y$.

From the first, $y^2 + x - 7 + \sqrt{3y^2 + 2x - 11} = 2y$;

$$\therefore\ 2y^2 + 2x - 14 + 2\sqrt{3y^2 + 2x - 11} = 4y;$$

$$\therefore\ (3y^2 + 2x - 11) + 2\sqrt{3y^2 + 2x - 11} = y^2 + 4y + 3;$$

$\therefore$ completing the square and extracting the square roots, &c. we get

$$\sqrt{3y^2 + 2x - 11} = y + 1,$$

$$\text{and } \therefore\ 3y^2 + 2x - 11 = y^2 + 2y + 1,$$

$$\text{from which } x = 6 + y - y^2:$$

$$\text{hence } \sqrt{3y - x + 7} = y + 1,$$

$$\text{and } \therefore\ y + 1 = \frac{x+y}{x-y}:$$

$$\text{wherefore } xy = y^2 + 2y, \text{ and } x = y + 2:$$

$\therefore$ equating the two values of $x$, we have

$$y + 2 = 6 + y - y^2,$$

$$\therefore\ y^2 = 4 \text{ and } y = 2,$$

$$\text{and } \therefore\ x = y + 2 = 4.$$

Ex. 7. Given $x^{x+y} = y^{4m}$ and $y^{x+y} = x^m$, to find the values of $x$ and $y$.

From the first, $x = y^{\frac{4m}{x+y}}$, and from the second, $x = y^{\frac{x+y}{m}}$:

$$\text{whence we have } y^{\frac{4m}{x+y}} = y^{\frac{x+y}{m}},$$

$$\text{and } \therefore \frac{x+y}{m} = \frac{4m}{x+y},$$

$$\text{whence } (x+y)^2 = 4m^2, \text{ and } \therefore x+y = \pm 2m;$$

$$\therefore \text{ using the positive sign, we have } x = y^{\frac{x+y}{m}} = y^2,$$

$$\therefore y^2 + y = 2m;$$

$$\text{wherefore } y^2 + y + \frac{1}{4} = \frac{8m+1}{4},$$

$$\text{and } \therefore y = \frac{-1 \pm \sqrt{8m+1}}{2};$$

$$\therefore x = y^2 = \frac{4m+1 \mp \sqrt{8m+1}}{2}.$$

If the negative sign had been retained, then since

$$\frac{x+y}{m} = -2,$$

$$\text{we shall have } x = \frac{1}{y^2},$$

$$\text{whence } x+y = -2m \text{ becomes } \frac{1}{y^2} + y = -2m,$$

$$\text{and } \therefore y^3 + 2my + 1 = 0 \text{ or } y^3 + 2my = -1,$$

neither of which have we as yet learned to solve.

Ex. 8. Given $bz + cy = a$, $az + cx = b$ and $ay + bx = c$, to find the values of $x$, $y$ and $z$.

Multiplying the first equation by $a$, the second by $b$ and the third by $c$, we have

$$abz + acy = a^2,$$

$$abz + bcx = b^2,$$

$$acy + bcx = c^2;$$

$\therefore$ by addition, $2abz + 2acy + 2bcx = a^2 + b^2 + c^2$:

but $2abz + 2acy = 2a^2$, from the first;

$\therefore$ by subtraction, $2bcx = b^2 + c^2 - a^2$,

$$\text{and } \therefore\ x = \frac{b^2 + c^2 - a^2}{2bc}:$$

$$\text{similarly } y = \frac{a^2 + c^2 - b^2}{2ac} \text{ and } z = \frac{a^2 + b^2 - c^2}{2ab}.$$

174. From what has been said in (171), it manifestly follows that the values of the unknown quantities may be expressed in terms of the known magnitudes, when the numbers of unknown quantities and equations involving them are equal; in other words, that $n$ equations are necessary and sufficient for the determination of $n$ unknown quantities.

When $n+m$ equations are proposed to determine the values of only $n$ quantities supposed unknown, any $m$ of these equations may clearly be dispensed with, as they will be unnecessary if they lead to the same results as the rest, and point out some inconsistency in the proposed equations if they do not.

If the number of equations be only $n-m$, and the values of $n$ quantities be required, it follows from the article above referred to, that, after elimination, the final equation will involve $n-(n-m-1)$ or $m+1$ unknown quantities, whose values can therefore be exhibited only in terms of one another and the known coefficients: these cases constitute what is called the

*Indeterminate Analysis*, which will be treated of in the second part of the work.

To illustrate what was said in the former part of this article, we will introduce the following examples which are capable of solution by means of the principles already explained.

Ex. 1. Given the $n$ following equations: viz.

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \text{\&c.} + a_n x_n = 0,$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 + \text{\&c.} + b_n x_n = 0,$$
$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \text{\&c.} + c_n x_n = 0,$$
$$\text{\&c.} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{and } k_1 x_1^m + k_2 x_2^m + k_3 x_3^m + \text{\&c.} + k_n x_n^m = K;$$

to find the values of the $n$ unknown quantities

$$x_1, \ x_2, \ x_3, \ \text{\&c. } x_n.$$

Here it is obvious that, by means of the $n-1$ former equations, each of the unknown quantities $x_2$, $x_3$, &c. $x_n$, may be expressed in terms of the remaining one $x_1$, as has been done in some of the preceding examples; let therefore

$$x_2 = l_2 x_1, \ x_3 = l_3 x_1, \ \text{\&c.} = \text{\&c.}, \ x_n = l_n x_1;$$

then by the substitution of these values in the last equation, we obtain

$$k_1 x_1^m + k_2 l_2^m x_1^m + k_3 l_3^m x_1^m + \text{\&c.} + k_n l_n^m x_1^m = K,$$

$$\text{whence } x_1 = \sqrt[m]{\frac{K}{k_1 + k_2 l_2^m + k_3 l_3^m + \text{\&c.} + k_n l_n^m}},$$

from which may be immediately derived the values of the remaining quantities $x_2$, $x_3$, &c. $x_n$.

Ex. 2. Given the $n$ equations following: viz.

$$a_1x_1 + a_2x_2 + a_3x_3 + \&c. + a_nx_n = A,$$
$$b_1x_1 + b_2x_2 + b_3x_3 + \&c. + b_nx_n = B,$$
$$c_1x_1 + c_2x_2 + c_3x_3 + \&c. + c_nx_n = C,$$
$$\&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

to find the $n$ unknown quantities $x_1$, $x_2$, $x_3$, &c. $x_n$.

Multiplying the members of the first, second, third, &c. equations by the quantities 1, $p$, $q$, &c. respectively, we shall have

$$a_1x_1 + a_2x_2 + a_3x_3 + \&c. + a_nx_n = A,$$
$$pb_1x_1 + pb_2x_2 + pb_3x_3 + \&c. + pb_nx_n = pB,$$
$$qc_1x_1 + qc_2x_2 + qc_3x_3 + \&c. + qc_nx_n = qC,$$
$$\&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\therefore$ by the addition of these quantities in vertical rows, we get

$$(a_1 + pb_1 + qc_1 + \&c.)\,x_1 + (a_2 + pb_2 + qc_2 + \&c.)\,x_2$$
$$+ (a_3 + pb_3 + qc_3 + \&c.)\,x_3 + \&c. + (a_n + pb_n + qc_n + \&c.)\,x_n$$
$$= A + pB + qC + \&c.$$

now if the coefficients of $x_2$, $x_3$, &c. $x_n$, in this equation be each assumed $=0$, we shall have

$$x_1 = \frac{A + pB + qC + \&c.}{a_1 + pb_1 + qc_1 + \&c.},$$

where the values of $p$, $q$, &c. may evidently be determined by means of the $n-1$ equations,

$$a_2 + b_2p + c_2q + \&c. = 0,$$
$$a_3 + b_3p + c_3q + \&c. = 0,$$
$$\&c. \ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n + b_np + c_nq + \&c. = 0,$$

as pointed out in the last example: and similarly of the rest of the unknown quantities $x_2$, $x_3$, &c. $x_n$.

Ex. 3. Let there be given the $n$ following equations:

$$a_1 x_1 x_2 + b_1 x_1 + c_1 x_2 + d_1 = 0,$$
$$a_2 x_2 x_3 + b_2 x_2 + c_2 x_3 + d_2 = 0,$$
$$a_3 x_3 x_4 + b_3 x_3 + c_3 x_4 + d_3 = 0,$$
&c.............................
$$a_n x_n x_1 + b_n x_n + c_n x_1 + d_n = 0;$$

to find the values of the $n$ unknown quantities

$$x_1,\ x_2,\ x_3,\ \&c.\ x_n.$$

From the first equation we obtain

$$x_2 = -\frac{b_1 x_1 + d_1}{a_1 x_1 + c_1};$$

wherefore, if we substitute this value of $x_2$ in the second equation, we shall be enabled to find $x_3$ in terms of $x_1$: and continuing this process, we shall at length find $x_n$ in the form

$$\frac{A x_1 + B}{C x_1 + D}:$$

and this being substituted in the last equation gives

$$a_n \frac{A x_1^2 + B x_1}{C x_1 + D} + b_n \frac{A x_1 + B}{C x_1 + D} + c_n x_1 + d_n = 0,$$

a quadratic, from which $x_1$ may be determined; and thence each of the remaining quantities $x_2$, $x_3$, &c. $x_n$ will become known.

### *Problems dependent upon simultaneous Equations.*

175. A proper number of the letters $x$, $y$, $z$, &c. being assumed to represent the magnitudes that are required, and the

problem being translated into algebraical language, according to the directions given and exemplified in (152), the number of independent equations thence arising will be equal to the number of unknown quantities, if the problem be determinate in its nature; and consequently, by the application of the methods of resolving such equations just given, we shall readily arrive at known values for these quantities, and thereby at the solution of the problem.

Ex. 1. Divide the number 9 into two parts so that five times the greater may exceed six times the less by 1.

Let $x$ and $y$ represent the two parts, whereof $x$ is the greater and $y$ the less: then we have the equations

$$x+y=9 \text{ and } 5x-6y=1,$$

which express the conditions of the problem algebraically:

also, from the former of these, $x=9-y$; wherefore, by substitution in the latter we get

$$45-11y=1;$$

$$\therefore\ 11y=44 \text{ and } \therefore\ y=4;$$

$$\therefore\ x=9-y=9-4=5:$$

so that the parts required are 5 and 4.

Ex. 2. In a certain employment nine men and seven women receive together £3. 11*s*. 2*d*. for their wages, and it is found that seven men receive 19*s*. 8*d*. more than five women: required the wages of each.

Let $x$ and $y$ represent the wages of each man and woman respectively in pence: then, by the question, we have

$$9x+7y=\text{£}3.\ 11s.\ 2d.=854d,$$

$$\text{and } 7x-5y=\quad 19s.\ 8d.=236d:$$

$$\text{from the former, } x=\frac{854-7y}{9},$$

$$\text{and from the latter } x = \frac{236+5y}{7};$$

whence we have

$$\frac{854-7y}{9} = \frac{236+5y}{7},$$

$$\text{and } \therefore\ 5978 - 49y = 2124 + 45y;$$

$$\therefore\ 94y = 3854,$$

and $y = 41d. = 3s.\ 5d,$ each woman's pay;

$$\text{and } \therefore\ x = \frac{854-7y}{9} = \frac{854-287}{9} = \frac{567}{9} = 63d. = 5s.\ 3d,$$

the pay of each man.

Ex. 3. Seven times the greater of two numbers and five times the less make 282, and three times the square of the former exceeds seven times the square of the latter by 1700: find them.

Assuming $x$ and $y$ to denote the required numbers, we shall evidently have $7x + 5y = 282$ and $3x^2 - 7y^2 = 1700$:

now from the first, $y = \dfrac{282 - 7x}{5}$, and $\therefore$ by substitution in the second we have

$$3x^2 - \frac{7}{25}(282 - 7x)^2 = 1700,$$

from which, by reduction, there results

$$268x^2 - 27636x = -599168;$$

$$\text{whence } x = 31;$$

$$\text{and } \therefore\ y = \frac{282-217}{5} = \frac{65}{5} = 13:$$

that is, the required numbers are 31 and 13.

Ex. 4. A tradesman in purchasing a piece of stuff, finds that if he had bought $a$ yards more at $b$ pence a yard less, he would have paid the same sum: but if he had bought $c$ yards more at $d$ pence a yard less, his payment would have been $e$ pence less: required the number of yards and the price per yard.

Let $x =$ the number of yards,

and $y =$ the price per yard;

then $xy =$ the price in pence:

also, $(x+a)(y-b) = xy$, by the question;

$\therefore\ ay - bx = ab$:

again, $(x+c)(y-d) = xy - e$, by the question;

$\therefore\ cy - dx = cd - e$:

from the former, $acy - bcx = abc$:

from the latter, $acy - adx = acd - ae$;

$\therefore$ by subtraction, $(ad - bc)x = abc - acd + ae$,

and $x = \dfrac{a(bc - cd + e)}{ad - bc}$, the number of yards;

$$\therefore\ y = \frac{ab + bx}{a} = b + \frac{b(bc - cd + e)}{ad - bc}$$

$$= \frac{b(ad - cd + e)}{ad - bc}, \text{ the price per yard.}$$

In order that this solution may accord with the enunciation of the problem, it is obvious that $bc - cd + e$ and $ad - cd + e$ must both be positive or both negative according as $ad$ is greater or less than $bc$.

If, however, it happen that $\dfrac{a}{c} = \dfrac{b}{d}$, then must we have also $bc - cd + e = 0$ and $ad - cd + e = 0$, so that $x$ and $y$ assume the intermediate form $\dfrac{0}{0}$.

but in this case our equations become

$$ay - cx = ab,$$

$$\text{and } cy - \frac{bc}{a}x = \frac{bc^2}{a} - e, \text{ since } d = \frac{bc}{a}:$$

$$\therefore\ acy - bcx = abc,$$

$$\text{and } acy - bcx = bc^2 - ae,$$

the former members of which being identical independently of any particular values of the unknown quantities, the latter must be so likewise, so that the number of independent equations is insufficient for the solution.

Ex. 5. Required two magnitudes whose sum is $a$, and the sum of whose fourth powers is $b^4$.

Let $x$ and $y$ represent the two magnitudes: then we have

$$x + y = a \text{ and } x^4 + y^4 = b^4, \text{ by the problem}:$$

$$\text{from the first, } x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = a^4,$$

$$\text{from the second, } x^4 \qquad\qquad\qquad\qquad + y^4 = b^4;$$

$$\therefore\ xy(2x^2 + 3xy + 2y^2) = \frac{a^4 - b^4}{2}:$$

$$\text{but since } x^2 + 2xy + y^2 = a^2,$$

$$\text{we have } 2x^2 + 3xy + 2y^2 = 2a^2 - xy;$$

$\therefore$ by substitution and reduction, we obtain

$$x^2y^2 - 2a^2xy + a^4 = \frac{a^4 + b^4}{2},$$

$$\text{whence } xy = a^2 \pm \sqrt{\frac{a^4 + b^4}{2}}:$$

$$\text{also, } x^2 + 2xy + y^2 = a^2,$$

$$\text{and } 4xy \quad = 4a^2 \pm 4\sqrt{\frac{a^4 + b^4}{2}},$$

$$\therefore\ x^2 - 2xy + y^2 = -3a^2 \mp 4\sqrt{\frac{a^4+b^4}{2}},$$

$$\text{and } x - y = \pm\sqrt{-3a^2 \mp 4\sqrt{\frac{a^4+b^4}{2}}};$$

whence we immediately obtain

$$x = \frac{a \pm \sqrt{-3a^2 \mp 4\sqrt{\frac{a^4+b^4}{2}}}}{2},$$

$$\text{and } y = \frac{a \mp \sqrt{-3a^2 \mp 4\sqrt{\frac{a^4+b^4}{2}}}}{2}:$$

which will always satisfy the proposed equations, but can be expressed in possible terms only when the lower signs of the quantity under the vinculum are used, or when

$$xy = a^2 - \sqrt{\frac{a^4+b^4}{2}}.$$

This will also be manifest from the circumstance that $x^2 + y^2$ must in such cases be a positive quantity.

Ex. 6. To find two magnitudes whose product is $a$, and the difference of their squares $2b$.

Taking $x$ and $y$ to denote the magnitudes required, we have

$$xy = a \text{ and } x^2 - y^2 = 2b:$$

$$\text{now } x^2 - y^2 = 2b \text{ and } \therefore\ 2\sqrt{-1}\,xy = 2a\sqrt{-1};$$

$\therefore$ by addition and subtraction, arise the equations,

$$x^2 + 2\sqrt{-1}\,xy - y^2 = 2b + 2a\sqrt{-1},$$

$$x^2 - 2\sqrt{-1}\,xy - y^2 = 2b - 2a\sqrt{-1}:$$

whence by evolution, we obtain

$$x + y\sqrt{-1} = \sqrt{2b + 2a\sqrt{-1}}$$

$$x - y\sqrt{-1} = \sqrt{2b - 2a\sqrt{-1}};$$

and from these equations, there result

$$x = \tfrac{1}{2}\left\{\sqrt{2b + 2a\sqrt{-1}} + \sqrt{2b - 2a\sqrt{-1}}\right\},$$

$$\text{and } y = \frac{1}{2\sqrt{-1}}\left\{\sqrt{2b + 2a\sqrt{-1}} - \sqrt{2b - 2a\sqrt{-1}}\right\}.$$

These values of $x$ and $y$ which may easily be proved to satisfy the equations above given, have no signification in their present form, and the method of solution will be entirely useless unless we can get rid of the imaginary symbol $\sqrt{-1}$, and prove them to be real. This might be done by effecting the operations indicated by the radical signs; but real results may be otherwise obtained, for

$$\frac{x}{y} = \sqrt{-1}\left\{\frac{\sqrt{2b + 2a\sqrt{-1}} + \sqrt{2b - 2a\sqrt{-1}}}{\sqrt{2b + 2a\sqrt{-1}} - \sqrt{2b - 2a\sqrt{-1}}}\right\}$$

$$= \sqrt{-1}\left\{\frac{4b + 4\sqrt{b^2 + a^2}}{4a\sqrt{-1}}\right\} = \frac{b + \sqrt{a^2 + b^2}}{a},$$

which combined with the equation $xy = a$, gives immediately

$$x^2 = b + \sqrt{a^2 + b^2} \text{ and } \therefore\ x = \pm\sqrt{b + \sqrt{a^2 + b^2}};$$

$$\therefore\ y = \frac{a}{x} = \pm\frac{a}{\sqrt{b + \sqrt{a^2 + b^2}}}.$$

Ex. 7. To find two quantities whose product shall be $a^2$, and the difference of their cubes equal to $m$ times the cube of their difference.

Let $x+y$ and $x-y$ denote the two quantities: then by the problem, we have

$$(x+y)\times(x-y)=a^2,$$

$$\text{and } (x+y)^3-(x-y)^3=8my^3:$$

$$\text{that is, } x^2-y^2=a^2,$$

$$\text{and } 6x^2y+2y^3=8my^3:$$

$$\therefore\ 3x^2+y^2=4my^2:$$

$$\text{also } 3x^2-3y^2=3a^2\cdot$$

$$\therefore \text{ by subtraction, } 4y^2=4my^2-3a^2,$$

$$\therefore\ 4(m-1)y^2=3a^2 \text{ and } y=\pm\frac{a}{2}\sqrt{\frac{3}{m-1}}:$$

$$\text{whence } x^2=a^2+y^2=a^2+\frac{3a^2}{4(m-1)}=\frac{a^2}{4}\left\{\frac{4m-1}{m-1}\right\},$$

$$\text{and } x=\pm\frac{a}{2}\sqrt{\frac{4m-1}{m-1}}:$$

and the quantities required are

$$x+y=\pm\frac{a}{2}\left\{\frac{\sqrt{4m-1}+\sqrt{3}}{\sqrt{m-1}}\right\},$$

$$\text{and } x-y=\pm\frac{a}{2}\left\{\frac{\sqrt{4m-1}-\sqrt{3}}{\sqrt{m-1}}\right\}.$$

Ex. 8. Required two magnitudes whose product is equal to the difference of their squares, and the sum of whose squares is equal to the difference of their cubes.

Let $x$ and $xy$ represent the two magnitudes:

$$\text{then } x^2y=x^2y^2-x^2 \quad\text{or}\quad y=y^2-1,$$

$$\text{and } x^2y^2+x^2=x^3y^3-x^3 \quad\text{or}\quad y^2+1=xy^3-x:$$

from the first equation, $y^2 - y = 1$, we find immediately

$$y = \frac{1 \pm \sqrt{5}}{2}:$$

and from the second, we have

$$x = \frac{y^2+1}{y^3-1} = \frac{y+2}{2y}, \text{ by substitution,}$$

$$= \frac{1}{2} + \frac{1}{y} = \frac{1}{2} + \frac{2}{1 \pm \sqrt{5}}$$

$$= \frac{1 \pm \sqrt{5} + 4}{2(1 \pm \sqrt{5})} = \frac{5 \pm \sqrt{5}}{2(1 \pm \sqrt{5})}:$$

wherefore the required magnitudes are

$$x = \frac{5 \pm \sqrt{5}}{1 \pm \sqrt{5}} = \frac{1}{2}\sqrt{5},$$

$$\text{and } xy = \frac{1}{2}\sqrt{5} \times \left(\frac{1 \pm \sqrt{5}}{2}\right) = \frac{1}{4}(\sqrt{5} \pm 5):$$

and from these results it is obvious that no *rational* magnitudes whatever possess the specified property.

Ex. 9. To find two numbers whose sum, product and the sum of whose squares are equal to each other.

Let $x$ and $y$ represent the required numbers, then will

$$x + y = xy,$$

$$\text{and } xy = x^2 + y^2:$$

$$\text{from the first, } x = \frac{y}{y-1},$$

$$\text{and from the second, } x = \frac{y \pm y\sqrt{-3}}{2}:$$

whence there arises the equation

$$\frac{1}{y-1}=\frac{1\pm\sqrt{-3}}{2},$$

$$\text{or } y-1=\frac{2}{1\pm\sqrt{-3}}=\frac{1\mp\sqrt{-3}}{2}:$$

$$\therefore\ y=\frac{3\mp\sqrt{-3}}{2}:$$

$$\text{and } x=\frac{y}{y-1}=\frac{3\pm\sqrt{-3}}{2}:$$

which results, though they satisfy the algebraical expressions, point out at the same time the impossibility of finding two *numbers* which will answer the conditions of the problem.

Ex. 10. Seven ducks and eight teal cost 27*s.* 4*d.*, eight ducks and five widgeons 22*s.* 8*d.*, and nine teal and seven widgeons 24*s.* 4*d.*: required the price of each duck, teal and widgeon.

If $x$, $y$ and $z$ be assumed to represent the prices in pence of a duck, teal and widgeon respectively, we shall have

$$7x+8y=328,$$

$$8x+5z=272,$$

$$9y+7z=292;$$

$\therefore$ from the first, $56x+64y=2624$, by multiplying by 8,

and from the second, $56x+35z=1904$, by multiplying by 7;

$\therefore$ by subtraction, $64y-35z=720$:

also, from the third, $45y+35z=1460$, by multiplying by 5;

$\therefore$ by addition, $109y=2180$,

$$\therefore\ y=\frac{2180}{109}=20d., \text{ the price of each teal;}$$

$$\therefore\ x = \frac{328-8y}{7} = \frac{328-160}{7} = \frac{168}{7} = 24d.,\ \text{of each duck;}$$

$$\text{and } z = \frac{272-8x}{5} = \frac{272-192}{5} = \frac{80}{5} = 16d.,\ \text{of each widgeon.}$$

Ex. 11. *A* and *B* together can perform a piece of work ($a$) in 8 days, *A* and *C* together in 9 days, and *B* and *C* together in 10 days: how many days will it take each person alone to perform the work?

Let $x$, $y$ and $z$ stand for the parts of the work performed by *A*, *B* and *C* respectively in one day: then, by the question,

$$8x + 8y = a,$$

$$9x + 9z = a,$$

$$10y + 10z = a;$$

$$\therefore\ x + y = \frac{a}{8},\ x + z = \frac{a}{9} \text{ and } y + z = \frac{a}{10}:$$

$$\text{whence } y - z = \frac{a}{8} - \frac{a}{9} = \frac{a}{72},$$

$$\text{also, } y + z = \frac{a}{10},$$

$$\therefore\ y = \frac{41a}{720},\ \text{and } z = \frac{31a}{720}:$$

$$\text{and } x = \frac{a}{8} - y = \frac{49a}{720}:$$

hence *A* can perform the work in $a \div \frac{49}{720}a = \frac{720}{49} = 14\frac{34}{49}$ days:

similarly *B* and *C* can do it in $17\frac{23}{41}$ days and $23\frac{7}{31}$ days respectively.

From the solution, it is evident that the results are entirely independent of the magnitude of the whole work.

Ex. 12. To find three magnitudes, when the quotients arising from dividing the products of every two by the one remaining are $a$, $b$ and $c$.

Let $x$, $y$ and $z$ denote the magnitudes required: then the conditions of the question give

$$\frac{xy}{z}=a, \quad \frac{xz}{y}=b \text{ and } \frac{yz}{x}=c:$$

from the first and second, we get

$$ab=\frac{xy}{z}\times\frac{xz}{y}=x^2 \text{ and } \therefore\ x=\pm\sqrt{ab};$$

from the first and third, we obtain

$$ac=\frac{xy}{z}\times\frac{yz}{x}=y^2 \text{ and } \therefore\ y=\pm\sqrt{ac};$$

and from the second and third, we find

$$bc=\frac{xz}{y}\times\frac{yz}{x}=z^2 \text{ and } \therefore\ z=\pm\sqrt{bc}.$$

Ex. 13. Three persons $A$, $B$ and $C$ possess certain sums of money, such that if $A$ receive half of the sums of $B$ and $C$ he will then have £$a$: if $B$ receive one third of the sums of $A$ and $C$ he will then have £$b$: and if $C$ receive one fourth of the sums of $A$ and $B$ he will then be possessed of £$c$: required the sum originally possessed by each.

Let $x$, $y$ and $z$ denote the required sums; then by the conditions of the problem, we have

$$x+\frac{y+z}{2}=a, \text{ or } 2x+y+z=2a,$$

$$y+\frac{x+z}{3}=b, \text{ or } 3y+x+z=3b,$$

$$z+\frac{x+y}{4}=c, \text{ or } 4z+x+y=4c:$$

subtracting the second equation from the first, we obtain

$$x-2y=2a-3b;$$

subtracting the third equation from four times the first, we get

$$7x+3y=8a-4c:$$

whence we have now the equations

$$x-2y=2a-3b,$$

$$\text{and } 7x+3y=8a-4c:$$

from the former of these, $7x-14y=14a-21b$,

and from the latter, $7x+3y=8a-4c$;

$\therefore$ by subtraction, $17y=21b-4c-6a$

$$\text{and } y=\frac{21b-4c-6a}{17}:$$

$$\text{whence } x=2a-3b+2y=\frac{22a-9b-8c}{17},$$

$$\text{and } z=2a-2x-y=\frac{20c-3b-4a}{17}.$$

176. We shall conclude this Chapter with the investigation and exemplification of the Arithmetical rules of *Single* and *Double Position*, which, in some cases, are made to supersede the use of Algebra.

(1.) In Single Position are considered those questions wherein the result of the operations upon the unknown quantity is always some multiple, part, or parts of the quantity itself, and the applicability of the rule is entirely decided by this circumstance.

Let $x$ be a number required which is to undergo such operations that the result is $a$; also, let $s$ be a *supposed* number which, by the same operations, gives a result $b$: then by the nature of the case we have

$$\frac{x}{a} = \frac{s}{b}, \text{ and } \therefore\ x = \frac{a}{b}s,$$

which enunciated at length is the rule.

Ex. Find a number which being increased by its fourth and seventh parts become 39.

Suppose the number to be 24; then its fourth and seventh parts are 6 and $3\frac{3}{7}$, so that the whole sum proposed becomes

$$24 + 6 + 3\tfrac{3}{7} \text{ or } 33\tfrac{3}{7},$$

whereas it ought to have been 39: hence, according to the expression just investigated, we have the required number

$$= \frac{39}{33\frac{3}{7}} \times 24 = \frac{273}{234} \times 24 = \frac{7}{6} \times 24 = 28,$$

which obviously answers the condition of the question.

(2.) In Double Position the result is no longer a certain multiple, part, or parts of the quantity itself, but involves further the addition or subtraction of some given magnitude, and the rule will be useless unless this is the case.

Let $x$ represent the number sought to satisfy the condition

$$ax + b = c,$$

where $a$, $b$ and $c$ are known magnitudes; and suppose $s$ and

$s'$ when substituted for $x$ not to fulfil the condition, but to give the errors $e$ and $e'$, both in excess; then we have

$$as+b=c+e \text{ and } as'+b=c+e';$$

$$\therefore\ a(s-x)=e \text{ and } a(s'-x)=e':$$

$$\text{whence } \frac{s-x}{s'-x}=\frac{e}{e'},$$

$$\text{and } \therefore\ x=\frac{es'-e's}{e-e'} \text{ or } =\frac{e's-es'}{e'-e},$$

which expressed in words is the rule.

If the errors be both in defect, the result is the same; but if one of the errors be in excess and the other in defect, the result will manifestly be $x=\dfrac{es'+e's}{e+e'}$.

Ex. 1. What number is that which being divided by 9 and the quotient diminished by 3, three times the remainder shall be 30?

Let 144 be the number $s$; then will the result of the operation expressed in the question

$$=3\left(\frac{144}{9}-3\right)=39,$$

so that the first error $e=39-30=9$:

again, let 126 be the number $s'$; then will the result, as before,

$$=3\left(\frac{126}{9}-3\right)=33,$$

so that the second error $e'=33-30=3$:

$$\therefore \text{ the number required } =\frac{es'-e's}{e-e'}=\frac{9.126-3.144}{9-3}$$

$$=\frac{1134-432}{6}=\frac{702}{6}=117,$$

which on trial will be found to fulfil the condition.

Ex. 2. *A* and *B* have both the same income; *A* saves one-fifth of his, but *B*, by spending £50. a year more than *A*, at the end of four years finds himself £100. in debt: what is the annual receipt and expenditure of each?

Suppose £150. to be the income of each;

then £30. = sum saved by *A*, and ∴ £120. = sum spent by him:

$$\therefore\ £120. + £50. = £170. = \text{annual expenditure of } B:$$

whence we have $(170 - 150) \times 4 = £80.$ = the debt incurred by *B* in four years: and therefore the first error is $100 - 80 = 20$ in defect:

again, let £100. be the income of each:

then £20. = sum saved by *A*, and ∴ £80. = sum spent by him:

$$\therefore\ £80. + £50. = £130. = \text{annual expenditure of } B:$$

∴ as before, $(130 - 100) \times 4 = £120.$ = the debt incurred by *B* in four years: consequently the second error is $120 - 100 = 20$ in excess:

$$\text{whence we have the income of each} = \frac{20 \,.\, 100 + 20 \,.\, 150}{20 + 20} = £125:$$

$$\therefore\ A\text{'s annual expenditure} = 125 - \frac{1}{5}(125) = £100:$$

$$\text{and } B\text{'s annual expenditure} = 100 + 50 = £150.$$

For an extensive collection of examples connected with the subjects of this Chapter, the reader is referred to BLAND's *Algebraical Problems.*

# CHAP. VII.

*On the Method of Indeterminate Coefficients. On the Binomial Theorem. On the Multinomial Theorem. On the Exponential Theorem.*

## I. Indeterminate Coefficients.

177. Def. The method of *Indeterminate Coefficients* is a process by which the *Expansion* or *Developement* of algebraical expressions may be effected, by assuming for them a series of powers of one of the letters involved, combined with coefficients whose values are afterwards to be assigned in terms of the rest; and it is of most extensive utility, as will appear from the applications of it that occur in the subsequent pages.

178. If the equation

$$A + Bx + Cx^2 + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.,$$

wherein both members are continued at pleasure, hold good for all values that can possibly be assigned to $x$, then will the coefficients of the same powers of $x$ in both members be equal to one another:

that is, $A = a$, $B = b$, $C = c$, $D = d$, &c. = &c.

For, since, independently of any particular value of $x$,

$$A + Bx + Cx^2 + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.$$

assume $x = 0$, and $\therefore A = a$: wherefore there then remains

$$Bx + Cx^2 + Dx^3 + \&c. = bx + cx^2 + dx^3 + \&c.$$

$$\text{or } B + Cx + Dx^2 + \&c. = b + cx + dx^2 + \&c.$$

assume $x = 0$, and $\therefore B = b$:

and by a continuation of this mode of reasoning it may manifestly be demonstrated that

$$C = c, \quad D = d, \quad \&c. = \&c.$$

The truth of this proposition is further manifest from the circumstance, that if we transpose all the terms of the second side of the equation, we shall have

$$(A - a) + (B - b)\,x + (C - c)\,x^2 + (D - d)\,x^3 + \&c. = 0,$$

which, if the coefficients $A - a$, $B - b$, $C - c$, $D - d$, &c. of the different powers of $x$ were finite, could be satisfied only by the *roots* of the equation, and thus the generality essential to the expression would be entirely destroyed.

179. Cor. 1. If the equation

$$A + Bx = a + bx,$$

hold good for *two* particular values of $x$, then will

$$A = a \text{ and } B = b.$$

For, let $\alpha$ and $\beta$ be the two values of $x$, so that

$$A + B\alpha = a + b\alpha \text{ and } A + B\beta = a + b\beta;$$

$$\therefore \text{ by subtraction, } B(\alpha - \beta) = b(\alpha - \beta),$$

whence $B = b$, and therefore $A = a$.

Similarly, if $A + Bx + Cx^2 = a + bx + cx^2$, be true for *three* particular values of $x$, then will $A = a$, $B = b$ and $C = c$: and so on for any number of terms and correspondent values of $x$: and this will manifestly lead to the conclusion in the last article.

180. Cor. 2. If $A + Bx + Cx^2 + Dx^3 + \&c. = 0$, the values of $a$, $b$, $c$, $d$, &c. being each $= 0$, we shall have likewise

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 0, \quad \&c.$$

181. By means of the principles just explained, the operations of division and evolution may be generally effected: integral quantities may be resolved into factors; and fractions, whose denominators consist of more factors than one, may be decomposed into others of a more simple description. This will appear by the following examples.

Ex. 1. To divide $a+x$ by $1-bx$, let us assume

$$\frac{a+x}{1-bx}=A+Bx+Cx^2+Dx^3+\&c.;$$

$\therefore$ multiplying both sides by $1-bx$, we shall have

$$\begin{aligned} a+x &= A+Bx+Cx^2+Dx^3+\&c. \\ &\qquad -Abx-Bbx^2-Cbx^3-\&c. \\ &= A+(B-Ab)\,x+(C-Bb)\,x^2+(D-Cb)\,x^3+\&c. \end{aligned}$$

whence, equating the coefficients of the same powers of $x$ in both members, we have

$$A=a;$$

$$B-Ab=1, \quad \therefore\ B=1+Ab=1+ab;$$

$$C-Bb=0, \quad \therefore\ C=Bb=(1+ab)\,b;$$

$$D-Cb=0, \quad \therefore\ D=Cb=(1+ab)\,b^2;$$

&c.......................................

so that

$$\frac{a+x}{1-bx}=a+(1+ab)\,x+(1+ab)\,bx^2+(1+ab)\,b^2x^3+\&c.$$

which is the same as obtainable by actual division.

Ex. 2. Let it be required to find the expansion of

$$\frac{a+bx}{\alpha+\beta x+\gamma x^2}.$$

Assume $$\frac{a+bx}{\alpha+\beta x+\gamma x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$$

$$\begin{aligned}\therefore\ a+bx=Aa+Ba x&+Ca x^2+Da x^3+Ea x^4+\&c.\\ &+A\beta x+B\beta x^2+C\beta x^3+D\beta x^4+\&c.\\ &\qquad+A\gamma x^2+B\gamma x^3+C\gamma x^4+\&c.\end{aligned}$$

$$\begin{aligned}&=Aa+(Ba+A\beta)\,x+(Ca+B\beta+A\gamma)\,x^2\\ &+(Da+C\beta+B\gamma)x^3+(Ea+D\beta+C\gamma)\,x^4+\&c.\end{aligned}$$

whence, equating the corresponding coefficients as before, we have

$$Aa=a,\quad \therefore\ A=\frac{a}{a};$$

$$Ba+A=b,\quad \therefore\ B=\frac{b}{a}-\frac{A\beta}{a}=\frac{ab-a\beta}{a^2};$$

$$Ca+B\beta+A\gamma=0,\quad \therefore\ C=-\frac{B\beta+A\gamma}{a}$$

$$=\frac{ab\beta-a\beta^2-aa\gamma}{a^3};\ \&c.$$

and it is obvious from the form of the coefficients of $x$, that any one of the succeeding quantities $D$, $E$, &c. may be found by multiplying the two immediately preceding it by $-\frac{\beta}{a}$ and $-\frac{\gamma}{a}$ respectively, and properly connecting the results: whence we shall have

$$\frac{a+bx}{a+\beta x+\gamma x^2}=\frac{a}{a}+\frac{ab-a\beta}{a^2}x+\frac{ab\beta-a\beta^2-aa\gamma}{a^3}x^2+\&c.$$

which, by means of the preceding remark, may be continued at pleasure.

Ex. 3. To extract the square root of $1+x^2$.

Assume $\sqrt{1+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\&c.$

$$\begin{aligned}\therefore\ 1+x^2 = A^2 &+ ABx + ACx^2 + ADx^3 + AEx^4 + \&c.\\ &+ ABx + B^2x^2 + BCx^3 + BDx^4 + \&c.\\ &\qquad + ACx^2 + BCx^3 + C^2x^4 + \&c.\\ &\qquad\qquad + ADx^3 + BDx^4 + \&c.\\ &\qquad\qquad\qquad + AEx^4 + \&c.\\ &\qquad\qquad\qquad\qquad + \&c.\end{aligned}$$

$\therefore$ equating the corresponding coefficients, we get

$$A^2=1, \text{ or } A=1;$$

$$2AB=0, \text{ or } B=0;$$

$$2AC+B^2=1, \text{ or } C=\frac{1-B^2}{2A}=\frac{1}{2};$$

$$2AD+2BC=0, \text{ or } D=-\frac{BC}{A}=0;$$

$$2AE+2BD+C^2=0, \text{ or } E=-\frac{2BD+C^2}{2A}=-\frac{1}{8};$$

&c.................................................

$$\text{whence } \sqrt{1+x^2}=1+\frac{1}{2}x^2-\frac{1}{8}x^4-\&c.:$$

and it may be observed that had we known the form of the expansion beforehand, the odd powers of $x$ might have been omitted in the assumption without altering the result.

Ex. 4. Let $x^3=Ax(x+1)(x+2)+Bx(x+1)+Cx+D$

$$=Ax^3+(3A+B)x^2+(2A+B+C)x+D;$$

then from (178) we shall have $A=1$;

$$3A+B=0, \text{ or } B=-3A=-3;$$

$$2A+B+C=0,$$

$$\therefore\ C=-2A-B=-2+3=1,$$

$$\text{and } D=0:$$

whence $x^3$ may be expressed, by factors, in the following form,

$$x(x+1)(x+2)-3x(x+1)+x.$$

Ex. 5. Assume $\dfrac{2a-x}{a^2-x^2}$ or $\dfrac{2a-x}{(a+x)(a-x)}=\dfrac{A}{a+x}+\dfrac{B}{a-x}$

$$=\frac{A(a-x)+B(a+x)}{a^2-x^2}$$

$$=\frac{(A+B)a-(A-B)x}{a^2-x^2};$$

$$\therefore\ 2a-x=(A+B)a-(A-B)x;$$

whence by (178), we have $A+B=2$ and $A-B=1$;

$$\therefore\ A=\frac{3}{2} \text{ and } B=\frac{1}{2};$$

$\therefore$ the fraction $\dfrac{2a-x}{a^2-x^2}$ is decomposed into $\dfrac{3}{2(a+x)}+\dfrac{1}{2(a-x)}$.

Ex. 6. Let it be required to decompose the fraction $\dfrac{1}{x^3-x^2-2x}$ into three others with simple factors for their denominators.

First, $x^3-x^2-2x=x(x^2-x-2)=x(x+1)(x-2)$, as will readily appear by (161):

assuming $\therefore \frac{1}{x^3 - x^2 - 2x} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x-2}$

$$= \frac{(A+B+C)x^2 - (2A+B-C)x - 2B}{(x+1)x(x-2)};$$

we shall, in consequence of the identity of the denominators, have

$$1 = (A+B+C)x^2 - (2A+B-C)x - 2B:$$

$$\therefore\ 2B = -1 \text{ and } B = -\frac{1}{2}:$$

$$\text{also, } 2A + B - C = 0 \text{ and } A + B + C = 0,$$

$$\therefore\ 2A - C = \frac{1}{2} \text{ and } A + C = \frac{1}{2};$$

$$\text{whence we obtain } A = \frac{1}{3} \text{ and } C = \frac{1}{6}:$$

$$\therefore\ \frac{1}{x^3 - x^2 - 2x} = \frac{1}{3(x+1)} - \frac{1}{2x} + \frac{1}{6(x-2)},$$

which is readily proved to be an identical equation.

182. This principle may be rendered still more general, and may be extended to indeterminate indices as well as indeterminate coefficients, so that if we have for every value of $x$,

$$Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} + \&c. = A'x^{\alpha'} + B'x^{\beta'} + C'x^{\gamma'} + \&c.$$

a similar process will lead to the conclusions that

$$\alpha = \alpha',\ \beta = \beta',\ \gamma = \gamma',\ \&c.;\ A = A',\ B = B',\ C = C',\ \&c.$$

## II. The Binomial Theorem.

183. Def. The *Binomial Theorem* is a general Algebraical Formula, by means of which any power or root of a quantity consisting of two terms may be expressed by a series

of simple quantities; and in one of its most general forms may be written

$$(a+x)^m$$

$$=a^m+m a^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3+\text{\&c.},$$

$$\text{or}\quad (a+x)^m=a^m\left(1+\frac{x}{a}\right)^m$$

$$=a^m\left\{1+\frac{m}{1}\left(\frac{x}{a}\right)+\frac{m(m-1)}{1.2}\left(\frac{x}{a}\right)^2+\frac{m(m-1)(m-2)}{1.2.3}\left(\frac{x}{a}\right)^3+\text{\&c.}\right\}$$

where the quantities $a$, $x$ and $m$ may be either positive or negative, integral or fractional.

We shall divide the proof of this theorem into the two following propositions:

(1) To determine the law of the formation of the indices, and the coefficient of the second term:

(2) To investigate the law of the formation of the succeeding coefficients:

and for the sake of simplicity, the binomial shall be represented by $1+v$ and its index by $m$.

184. *To determine the coefficient of the second term of the expansion of any power or root of* $1+v$, *and the law of the formation of the indices of* $v$.

First, let the index be a positive whole number $p$; then since by actual division

$$\frac{(1+v)^p-1}{(1+v)\ -1}=(1+v)^{p-1}+(1+v)^{p-2}+\text{\&c.}+(1+v)^2+(1+v)+1$$

$$=1+(1+v)+(1+v)^2+\text{\&c. to } p \text{ terms},$$

we shall have

$$(1+v)^p=1+v+v\left\{(1+v)+(1+v)^2+\text{\&c. to } (p-1) \text{ terms}\right\}$$

$$=1+pv+Bv^2+Cv^3+\text{\&c.},$$

since the first term of each of the binomials within the brackets, when expanded, is obviously $=1$: that is, the coefficient of the second term is the index $p$ and the indices of $v$ manifestly ascend regularly. This appears also from Ex. 6. of Art. 22.

Secondly, let the index be fractional and equal to $\frac{p}{q}$, and assume

$$(1+v)^{\frac{p}{q}} = 1 + Av + \&c.;\ \therefore\ (1+v)^p = (1 + Av + \&c.)^q:$$

wherefore, by the first case, we shall have

$$1 + pv + \&c. = 1 + qAv + \&c.$$

whence $qA = p$ and $\therefore\ A = \frac{p}{q}$, and the indices of $v$ must obviously ascend regularly:

$$\therefore\ (1+v)^{\frac{p}{q}} = 1 + \frac{p}{q}v + Bv^2 + Cv^3 + \&c.$$

Thirdly, let the index be any negative quantity either whole or fractional, represented by $-r$, then we have $(1+v)^{-r}$

$$= \frac{1}{(1+v)^r} = \frac{1}{1 + rv + \&c.} = 1 - rv + \&c.,$$

by actual division; and it is clear that the indices of $v$ will ascend regularly; therefore in this case also we shall have

$$(1+v)^{-r} = 1 - rv + Bv^2 + Cv^3 + \&c.$$

Hence, whether the index $m$ be positive or negative, integral or fractional, the coefficient of the second term is $m$ and the indices of $v$ ascend regularly, so that we may in every case assume

$$(1+v)^m = 1 + mv + Bv^2 + Cv^3 + \&c.$$

185. *To determine the law of the formation of the coefficients of the powers of* v *in the expansion of* $(1 + \mathrm{v})^{\mathrm{m}}$.

Let $(1+v)^m = 1 + mv + Bv^2 + Cv^3 + \&c. \qquad (\alpha)$

and for $v$ put $y+z$; then we shall have

$$\begin{aligned}(1+y+z)^m &= 1 + m(y+z) + B(y+z)^2 + C(y+z)^3 + \&c.\\ &= 1 + my + By^2 + Cy^3 + \&c.\\ &\quad + mz + 2Byz + 3Cy^2z + \&c.\\ &\quad + Bz^2 + 3Cyz^2 + \&c. \qquad (\beta)\\ &\quad + Cz^3 + \&c.\\ &\quad + \&c.\end{aligned}$$

again, by separating it into factors, we obviously have

$$(1+y+z)^m = \left\{(1+y)\left(1+\frac{z}{1+y}\right)\right\}^m = (1+y)^m\left(1+\frac{z}{1+y}\right)^m$$

$$= (1+y)^m\left\{1 + m\left(\frac{z}{1+y}\right) + B\left(\frac{z}{1+y}\right)^2 + C\left(\frac{z}{1+y}\right)^3 + \&c.\right\}$$

by $(\alpha)$,

$$= (1+y)^m + m(1+y)^{m-1}z + B(1+y)^{m-2}z^2 + C(1+y)^{m-3}z^3$$
$$+ \&c.$$

$$\begin{aligned}&= 1 + my + By^2 + Cy^3 + \&c.\\ &\quad + mz\,\{1 + (m-1)y + B'y^2 + C'y^3 + \&c.\}\\ &\quad + Bz^2\,\{1 + (m-2)y + B''y^2 + C''y^3 + \&c.\}\\ &\quad + Cz^3\,\{1 + (m-3)y + B'''y^2 + C'''y^3 + \&c.\}\\ &\quad + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

($B$, $C$, &c. becoming $B'$, $C'$, &c. $B''$, $C''$, &c. when the index $m$ becomes $m-1$, $m-2$, &c. respectively)

$$
\begin{array}{rl}
= 1 + my + & By^2 \quad + \quad Cy^3 + \&c. \\
+ mz + & m(m-1)yz + mB'y^2z + \&c. \\
+ & Bz^2 + (m-2)Byz^2 + \&c. \\
& \qquad\qquad + \quad C\ z^3 + \&c. \\
& \qquad\qquad\qquad\qquad + \&c.
\end{array}
\quad (\gamma)
$$

now, the series ($\beta$) and ($\gamma$) being identical, by equating the corresponding coefficients we shall obtain the following results: viz.

$$2B = m(m-1),$$

$$\therefore\ B = \frac{m(m-1)}{1.2}:$$

$$\therefore \text{ also, } B' = \frac{(m-1)(m-2)}{1.2};$$

$$3C = mB' = \frac{m(m-1)(m-2)}{1.2},$$

$$\therefore\ C = \frac{m(m-1)(m-2)}{1.2.3}:$$

$$\therefore \text{ likewise, } C' = \frac{(m-1)(m-2)(m-3)}{1.2.3};$$

$$\text{similarly, } 4D = mC' = \frac{m(m-1)(m-2)(m-3)}{1.2.3},$$

$$\therefore\ D = \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}:$$

and the process may obviously be continued as far as we please: whence we have

$$(1+v)^m = 1 + mv + \frac{m(m-1)}{1.2}v^2 + \frac{m(m-1)(m-2)}{1.2.3}v^3$$
$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}v^4 + \&c.$$

186. COR. 1. Hence, by separating $a+x$ into the factors $a$ and $1+\frac{x}{a}$, and then substituting $\frac{x}{a}$ in the place of $v$, we obtain

$$(a+x)^m = a^m\left(1+\frac{x}{a}\right)^m$$

$$=a^m\left\{1+\frac{m}{1}\left(\frac{x}{a}\right)+\frac{m(m-1)}{1.2}\left(\frac{x}{a}\right)^2+\frac{m(m-1)(m-2)}{1.2.3}\left(\frac{x}{a}\right)^3+\&c.\right\}$$

$$=a^m+ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3+\&c.:$$

where the quantities $a$, $x$ and $m$ may be positive or negative, integral or fractional, or indeed irrational.

If $m$ and $x$ be both positive, we shall have

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \&c.$$

If $m$ be positive and $x$ negative, we shall obtain

$$(a-x)^m = a^m - ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 - \&c.$$

If $m$ be negative and $x$ positive, we shall find

$$(a+x)^{-m} = a^{-m} - ma^{-m-1}x + \frac{m(m+1)}{1.2}a^{-m-2}x^2 - \&c.$$

$$=\frac{1}{a^m} - \frac{mx}{a^{m+1}} + \frac{m(m+1)x^2}{1.2.a^{m+2}} - \&c.$$

If $m$ and $x$ be both negative, we shall get

$$(a-x)^{-m} = a^{-m} + ma^{-m-1}x + \frac{m(m+1)}{1.2}a^{-m-2}x^2 + \&c.$$

$$=\frac{1}{a^m} + \frac{mx}{a^{m+1}} + \frac{m(m+1)x^2}{1.2.a^{m+2}} + \&c$$

Also, if the index be fractional and equal to $\pm\frac{p}{q}$, then will

$$(a\pm x)^{\frac{p}{q}}=a^{\frac{p}{q}}\pm\frac{p}{q}a^{\frac{p}{q}-1}x+\frac{p(p-q)}{1.2.q^2}a^{\frac{p}{q}-2}x^2\pm \&c.;$$

and

$$(a\pm x)^{-\frac{p}{q}}=a^{-\frac{p}{q}}\mp\frac{p}{q}a^{-\frac{p}{q}-1}x+\frac{p(p+q)}{1.2.q^2}a^{-\frac{p}{q}-2}x^2\mp \&c.$$

187. Cor. 2. If $A_1$, $A_2$, $A_3$, &c. be assumed to represent the first, second, third, &c. terms of the expansion, we may exhibit the theorem in another form.

For, $(a+x)^m=a^m+ma^{m-1}x+\dfrac{m(m-1)}{1.2}a^{m-2}x^2+\&c.$

$$=a^m+mA_1\frac{x}{a}+\frac{(m-1)}{2}A_2\frac{x}{a}+\frac{(m-2)}{3}A_3\frac{x}{a}+\&c.$$

by means of which, any term may easily be derived from that what immediately precedes it.

188. *To find the* n$^{\text{th}}$ *term, or the general term of the expansion of* $(1+v)^m$.

This may be determined by induction: for we have seen in article (185) that

the first term $=1$,

the second term $=mv$,

the third term $=\dfrac{m(m-1)}{1.2}v^2$,

the fourth term $=\dfrac{m(m-1)(m-2)}{1.2.3}v^3$, &c.;

whence, observing the connection subsisting among the numeral magnitudes, we shall obviously have

$$\text{the } n^{\text{th}} \text{ term} = \frac{m(m-1)(m-2).\&c.(m-n+2)}{1.2.3.\&c.(n-1)} v^{n-1}:$$

and the theorem, with its general term, may therefore be written

$$(1+v)^m = 1 + mv + \frac{m(m-1)}{1.2} v^2 + \frac{m(m-1)(m-2)}{1.2.3} v^3 + \&c.$$

$$+ \frac{m(m-1)(m-2).\&c.(m-n+2)}{1.2.3.\&c.(n-1)} v^{n-1} + \&c.:$$

and hence we shall have also the more general form

$$(a+x)^m$$

$$= a^m + m a^{m-1} x + \frac{m(m-1)}{1.2} a^{m-2} x^2 + \&c.$$

$$+ \frac{m(m-1)(m-2).\&c.(m-n+2)}{1.2.3.\&c.(n-1)} a^{m-n+1} x^{n-1} + \&c.$$

189. Cor. 1. If the index $m$ be a positive whole number and we suppose

$$m-n+2=0, \text{ or } n=m+2,$$

the $n^{\text{th}}$ and each succeeding term, involving zero as a factor, becomes $=0$, and consequently the series terminates after the $(n-1)^{\text{th}}$ or $(m+1)^{\text{th}}$ term: that is, the expansion of a binomial whose index is the positive integer $m$ contains $m+1$ terms, and the number of terms of the expansion will therefore be even or odd according as the index is odd or even.

Also, if $m$ be either fractional or negative, it is manifest that no one of the factors of the $n^{\text{th}}$ term can ever become $=0$; and consequently that the expansion will consist of a number

of terms indefinitely continued, so that the sign = which connects $(1+v)^m$ and its developement, must then be interpreted in the sense attached to it in (88).

190. Cor. 2. If the index $m$ be an even number, and therefore the number of terms be odd, the expansion of $(1+v)^m$ will have its middle term equal to

$$\frac{1.3.5.\&c.(m-1)}{1.2.3.\&c.\frac{1}{2}m}(2v)^{\frac{m}{2}}.$$

For, the middle term, which is the $(\frac{1}{2}m+1)^{\text{th}}$, is obviously

$$=\frac{m(m-1)(m-2).\&c.(\frac{1}{2}m+1)}{1.2.3.\&c.\frac{1}{2}m}v^{\frac{m}{2}};$$

and this, by multiplying the numerator and denominator by $1.2.3.\&c.\frac{1}{2}m$, becomes

$$=\frac{1.2.3.\&c.\frac{1}{2}m(\frac{1}{2}m+1).\&c.(m-2)(m-1)m}{(1.2.3.\&c.\frac{1}{2}m)^2}v^{\frac{m}{2}}$$

$$=\frac{\{1.3.5.\&c.(m-1)\}\times\{2.4.6.\&c.m\}}{(1.2.3.\&c.\frac{1}{2}m)^2}v^{\frac{m}{2}}$$

$$=\frac{\{1.3.5.\&c.(m-1)\}\times\{1.2.3.\&c.\frac{1}{2}m\}2^{\frac{m}{2}}}{(1.2.3.\&c.\frac{1}{2}m)^2}v^{\frac{m}{2}}$$

$$=\frac{1.3.5.\&c.(m-1)}{1.2.3.\&c.\frac{1}{2}m}(2v)^{\frac{m}{2}}.$$

191. Cor. 3. If the index $m$ be an odd number, and consequently the number of terms of the expansion of $(1+v)^m$ be even, there will obviously be two middle terms, the $\left(\frac{m+1}{2}\right)^{\text{th}}$ and $\left(\frac{m+3}{2}\right)^{\text{th}}$ from the beginning, which are therefore equal to

$$\frac{m(m-1)(m-2).\&c.\frac{1}{2}(m+3)}{1.2.3.\&c.\frac{1}{2}(m-1)}v^{\frac{m-1}{2}},$$

and $$\frac{m(m-1)(m-2).\&c.\frac{1}{2}(m+1)}{1.2.3.\&c.\frac{1}{2}(m+1)}v^{\frac{m+1}{2}};$$

and these may, as in the last corollary, be made to assume respectively, the forms

$$\frac{1.3.5.\&c.\, m}{1.2.3.\&c.\frac{1}{2}(m-1)}2^{\frac{m-1}{2}}v^{\frac{m-1}{2}} \text{ and } \frac{1.3.5.\&c.\, m}{1.2.3.\&c.\frac{1}{2}(m-1)}2^{\frac{m-1}{2}}v^{\frac{m+1}{2}}.$$

192. *If the index be a positive whole number, all the coefficients of the expansion of* $(1+v)^m$ *will be integral quantities.*

For, if the index $m = p + n - 1$, where $p$ is necessarily a positive whole number, we shall, by reversing the order of the factors, have the $n^{\text{th}}$ or general term

$$= \frac{(p+1)(p+2)(p+3).\&c.(p+n-1)}{1.2.3.\&c.(n-1)}:$$

but it is obvious that

$$\frac{(p+1)(p+2)(p+3).\&c.(p+n-1)}{1.2.3.\&c.(n-1)}$$

$$= \left\{1 + \frac{p}{n-1}\right\}\left\{\frac{(p+1)(p+2)(p+3).\&c.(p+n-2)}{1.2.3.\&c.(n-2)}\right\}$$

$$= \frac{(p+1)(p+2)(p+3).\&c.(p+n-2)}{1.2.3.\&c.(n-2)}$$

$$+ \frac{p(p+1)(p+2).\&c.(p+n-2)}{1.2.3.\&c.(n-1)}:$$

similarly, we shall have the expression

$$\frac{p(p+1)(p+2) \,.\, \&c. \,(p+n-2)}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-1)}$$

$$= \frac{p(p+1)(p+2) \,.\, \&c. \,(p+n-3)}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-2)}$$

$$+ \frac{(p-1)(p-2)(p-3) \,.\, \&c. \,(p+n-3)}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-1)};$$

&c. = &c.....................................

whence we infer that the general term will be integral for any value of $n$, provided it be so for the next inferior value and also when each of the factors of the numerator is diminished by any number less than $p$: but since the latter term at length becomes

$$= \frac{(n-1) \,.\, \&c. \; 3 \,.\, 2 \,.\, 1}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-1)} = 1,$$

it manifestly follows that if the expression be integral for any one value of $n$, it will necessarily be so for the next superior value: and the coefficient of the second term, being the index, is a whole number, therefore the coefficient of the third term is integral; therefore that of the fourth is a whole number, and so on: that is, the expression

$$\frac{(p+1)(p+2)(p+3) \,.\, \&c. \,(p+n-1)}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-1)},$$

or its equivalent

$$\frac{m(m-1)(m-2) \,.\, \&c. \,(m-n+2)}{1 \,.\, 2 \,.\, 3 \,.\, \&c. \,(n-1)},$$

is integral for every positive integral value of $m$, independently of the value of $n$ which *must* be integral; and consequently all the coefficients of the expanded binomial are whole numbers.

193. *On the same hypothesis, the coefficient of any term of the expansion reckoned from the end, is the same as the coefficient of the corresponding term reckoned from the beginning.*

Since, by (189), the whole number of terms of the expansion is $m+1$, it is evident that the $n^{\text{th}}$ term from the end is the $\{(m+1)-(n-1)\}^{\text{th}}$ or $(m-n+2)^{\text{th}}$ from the beginning: and therefore the coefficient of the $n^{\text{th}}$ term from the end is the coefficient of the $(m-n+2)^{\text{th}}$ term from the beginning: that is, the coefficient of the $n^{\text{th}}$ term from the end

$$= \frac{m(m-1)(m-2).\&c.\, n}{1.2.3.\&c.\,(m-n+1)}$$

$$= \frac{m(m-1)(m-2).\&c.\,(m-n+2)(m-n+1).\&c.\, n}{1.2.3.\&c.\,(n-1)\,n.\&c.\,(m-n+1)}$$

$$= \frac{m(m-1)(m-2).\&c.\,(m-n+2)}{1.2.3.\&c.\,(n-1)}:$$

which, as appears from (188), is also the coefficient of the $n^{\text{th}}$ term from the beginning: whence it follows, that if the index be a positive whole number, the corrosponding coefficients from the beginning and end are the same.

194. *To find expressions for the sums of the terms in the odd and even places of the expansion* of $(1+\mathrm{v})^{\mathrm{m}}$.

Since, by giving successively to $v$ a positive and negative sign, we have

$$(1+v)^m = 1 + mv + \frac{m(m-1)}{1.2}v^2 + \frac{m(m-1)(m-2)}{1.2.3}v^3 + \&c.$$

and

$$(1-v)^m = 1 - mv + \frac{m(m-1)}{1.2}v^2 - \frac{m(m-1)(m-2)}{1.2.3}v^3 + \&c.$$

$\therefore$ by addition and division by 2, we obtain

$$\frac{(1+v)^m+(1-v)^m}{2}$$

$$=1+\frac{m(m-1)}{1.2}v^2+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}v^4+\&c.$$

which is manifestly the sum of the terms in the odd places from the beginning:

and by subtraction and division by 2, we get

$$\frac{(1+v)^m-(1-v)^m}{2}=mv+\frac{m(m-1)(m-2)}{1.2.3}v^3+\&c.$$

which is obviously the sum of the terms in the even places.

195. Cor. 1. If in the expressions above deduced $v$ be made $=1$, we shall have

the sum of the *coefficients* of the terms in the odd places

$$=\frac{2^m}{2}=2^{m-1}$$

= the sum of the *coefficients* of the terms in the even places:

and hence it also follows that the sum of all the coefficients of any expanded binomial, whose index is $m$, is equal to

$$2\times 2^{m-1}=2^m.$$

This corollary is also evident from the expansions,

$$2^m=(1+1)^m=1+m+\frac{m(m-1)}{1.2}+\frac{m(m-1)(m-2)}{1.2.3}+\&c.$$

and

$$0=(1-1)^m=1-m+\frac{m(m-1)}{1.2}-\frac{m(m-1)(m-2)}{1.2.3}+\&c.$$

196. Cor. 2. Hence we may find also the sum of the series arising from multiplying the coefficients by the successive natural numbers 1, 2, 3, 4, &c.

For, $1+2m+3\dfrac{m(m-1)}{1.2}+4\dfrac{m(m-1)(m-2)}{1.2.3}+\text{\&c.}$

$$=1+m+\frac{m(m-1)}{1.2}+\frac{m(m-1)(m-2)}{1.2.3}+\text{\&c.}$$

$$+m+2\frac{m(m-1)}{1.2}+3\frac{m(m-1)(m-2)}{1.2.3}+\text{\&c.}:$$

now the former of these series manifestly $=(1+1)^m=2^m$,

and the latter $=m\left\{1+(m-1)+\dfrac{(m-1)(m-2)}{1.2}+\text{\&c.}\right\}$

$$=m(1+1)^{m-1}=m2^{m-1}:$$

wherefore the proposed sum $=2^m+m2^{m-1}=(m+2)\,2^{m-1}$.

Similarly, it may be demonstrated that

$$1-2m+3\frac{m(m-1)}{1.2}-4\frac{m(m-1)(m-2)}{1.2.3}+\text{\&c.}=0.$$

197. *To find the greatest term of the expansion of* $(1+v)^m$, *and also the greatest coefficient.*

Let $N$ and $N_1$ represent the $n^{\text{th}}$ and $(n+1)^{\text{th}}$ terms respectively; then we shall have from (188)

$$N=\frac{m(m-1)(m-2).\text{\&c.}(m-n+2)}{1.2.3.\text{\&c.}(n-1)}v^{n-1}, \text{ and}$$

$$N_1=\frac{m(m-1)(m-2).\text{\&c.}(m-n+2)(m-n+1)}{1.2.3.\text{\&c.}(n-1)n}v^n;$$

$$\text{whence } \frac{N_1}{N}=\frac{(m-n+1)v}{n}:$$

now, if, for any value of $n$, we have $N_1$ less than $N$, it will manifestly be so for every succeeding value of $n$: wherefore if the $n^{\text{th}}$ term be the greatest, we must have

$$\frac{(m-n+1)v}{n} \text{ less than } 1:$$

$$\therefore\ (m-n+1)v \text{ is less than } n;$$

$$\therefore\ (m+1)v \text{ is less than } n(v+1),$$

$$\text{whence } n \text{ is greater than } (m+1)\frac{v}{v+1}:$$

that is, the greatest term is that whose place is denoted by the whole number which is next greater than $(m+1)\dfrac{v}{v+1}$.

Also, if $v$ be assumed $=1$, so that the terms become equal to their corresponding coefficients, we shall have $n$ equal to the whole number next greater than

$$\frac{m+1}{2} \text{ or } \frac{m}{2}+\frac{1}{2}:$$

that is, the greatest coefficient is that whose place is expressed by $\dfrac{m}{2}+1$, if $m$ be an even whole number: and when $m$ is an odd integral quantity, the two greatest coefficients are equal to each other by (193), and the place of the former will manifestly be denoted by $\dfrac{m+1}{2}$.

198. Cor. 1. If the index be negative, it is manifest that $\dfrac{N_1}{N}$ will also be negative, and consequently in this case we must have $\dfrac{(m-n+1)v}{n}$ less than $-1$; whence $n$ is greater than

$(m+1)\dfrac{v}{v-1}$; and this, by assuming $v=1$, becomes indefinitely great, or in other words, the succeeding *coefficients* increase continually.

Also, since $n$ must be a positive whole number, the first term will be the greatest, should the expression $(m+1)\dfrac{v}{v-1}$ be negative.

199. Cor. 2. We have seen in article (197) that

$$\frac{N_1}{N}=\frac{(m-n+1)v}{n}=\left(\frac{m+1}{n}-1\right)v:$$

and in order to obtain the greatest and least values of this expression, we must make $n=1$ and $n=\infty$, which proves the extreme values of $\dfrac{N_1}{N}$ to be $mv$ and $-v$, so that the developement of $(1+v)^m$ is intermediate to the magnitudes

$$1+mv+m^2v^2+m^3v^3+\&c.$$

$$\text{and } 1-v+v^2-v^3+\&c.$$

each of these series being continued to $m+1$ terms, when $m$ is a positive whole number, and to infinity when it is not.

Hence also, the series after $n$ terms is of intermediate magnitude to the two series

$$Nv\left(\frac{m+1}{n}-1\right)+Nv^2\left(\frac{m+1}{n}-1\right)^2+Nv^3\left(\frac{m+1}{n}-1\right)^3+\&c.$$

$$\text{and } -Nv+Nv^2-Nv^3+\&c.$$

200. *The sum of the squares of the coefficients of the expansion of* $(1+v)^m$ *is equal to the coefficient of the middle term of the expansion of* $(1+v)^{2m}$.

Let $(1+v)^m = 1 + Av + Bv^2 + \&c. + Bv^{m-2} + Av^{m-1} + v^m$, the coefficients being the same from the beginning and end, $\therefore$ we have

$$\begin{aligned}(1+v)^{2m} = &\{1 + Av + Bv^2 + \&c. + Bv^{m-2} + Av^{m-1} + v^m\} \\ &\times \{v^m + Av^{m-1} + Bv^{m-2} + \&c. + Bv^2 + Av + 1\} \\ = &\ v^m + Av^{m+1} + Bv^{m+2} + \&c. \\ &\quad + Av^{m-1} + A^2v^m \quad + ABv^{m+1} + \&c. \\ &\qquad\qquad + Bv^{m-2} + ABv^{m-1} + B^2v^m + \&c. \\ &\qquad\qquad\qquad + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

now, the expansion of $(1+v)^{2m}$ containing $2m+1$ terms, it is obvious that its $(m+1)^{\text{th}}$ term will be equal to the sum of the terms of this latter expression involving $v^m$, which is

$$\{1^2 + A^2 + B^2 + C^2 + \&c.\}\, v^m:$$

whence we have the coefficient of the middle term of the expansion of $(1+v)^{2m}$ equal to the sum of the squares of the coefficients of the expansion of $(1+v)^m$.

201. Cor. Since the coefficient of the $(m+1)^{\text{th}}$ term of the expansion of $(1+v)^{2m} = \dfrac{2m\,(2m-1)\,(2m-2)\,.\,\&c.\,(m+1)}{1\,.\,2\,.\,3\,.\,\&c.\ m}$,

we have

$$1^2 + m^2 + \left\{\frac{m\,(m-1)}{1\,.\,2}\right\}^2 + \&c. = \frac{2m(2m-1)(2m-2)\,.\,\&c.\,(m+1)}{1\,.\,2\,.\,3\,.\,\&c.\ m}$$

$$= \frac{1\,.\,3\,.\,5\,.\,\&c.\ (2m-1)}{1\,.\,2\,.\,3\,.\,\&c.\ m}\,2^m = \frac{1\,.\,2\,.\,3\,.\,\&c.\ 2m}{(1\,.\,2\,.\,3\,.\,\&c.\ m)^2}.$$

202. *To find the sum of the products of every two contiguous coefficients of the expansion of* $(1+v)^m$.

Assume

$$(1+v)^m = 1 + Av + Bv^2 + \&c. + Bv^{m-2} + Av^{m-1} + v^m,$$

the coefficients from the beginning and end being the same, as appears from (193): hence, reversing the order of the terms, we shall have

$$(1+v)^m = v^m + Av^{m-1} + Bv^{m-2} + \&c. + Bv^2 + Av + 1:$$

whence multiplying the members of these two equations respectively together, we shall manifestly have in the expansion of $(1+v)^{2m}$ the coefficient of $v^{m-1}$ equal to

$$2\,\{1\,.\,A + A\,.\,B + B\,.\,C + \&c.\};$$

but the coefficient of $v^{m-1}$ in the expansion of $(1+v)^{2m}$ will by (188) be equal to

$$\frac{2m\,(2m-1)\,(2m-2)\,.\,\&c.\,(m+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(m-1)};$$

and from these there will obviously result

$$1\,.\,A + A\,.\,B + B\,.\,C + \&c. = \frac{1}{2}\left\{\frac{2m\,(2m-1)\,(2m-2)\,.\,\&c.\,(m+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(m-1)}\right\}.$$

Similarly, may be found the sum of the products of the first and third, second and fourth, &c. coefficients of $(1+v)^m$: and so on.

203. As an extension of the principle made use of in the last three articles, were we to assume

$$(1+v)^p = 1 + P_1 v + P_2 v^2 + P_3 v^3 + \&c. + P_p v^p,$$

$$(1+v)^q = 1 + Q_1 v + Q_2 v^2 + Q_3 v^3 + \&c. + Q_q v^q,$$

$$(1+v)^r = 1 + R_1 v + R_2 v^2 + R_3 v^3 + \&c. + R_r v^r,$$

&c. ............................................

and $(1+v)^{p+q+r+\&c.} = 1 + S_1 v + S_2 v^2 + S_3 v^3 + \&c.$

$$+ S_{p+q+r+\&c.}\, v^{p+q+r+\&c.};$$

since $(1+v)^{p+q+r+\&c.} = (1+v)^p \times (1+v)^q \times (1+v)^r \times$ &c.

we should, by effecting the multiplications of the latter members of these equations, and equating the coefficients of the same powers of $v$ in the equivalent expressions, be enabled to ascertain, were it of any utility, the existence of some curious connections and relations subsisting among the indices $p$, $q$, $r$, &c.

Thus, we should have the coefficient of the $n^{\text{th}}$ term of $(1+v)^{p+q}$ = the coefficient of $v^{n-1}$ in the product which arises from multiplying together the two expansions

$$1 + pv + \frac{p(p-1)}{1.2} v^2 + \&c. + \frac{p(p-1)(p-2).\&c.(p-n+2)}{1.2.3.\&c.(n-1)} v^{n-1}$$

$+$ &c. and

$$1 + qv + \frac{q(q-1)}{1.2} v^2 + \&c. + \frac{q(q-1)(q-2).\&c.(q-n+2)}{1.2.3.\&c.(n-1)} v^{n-1}$$

$+$ &c.

204. We have seen in a preceding article that $(1-v)^{-m}$

$$= 1 + mv + \frac{m(m+1)}{1.2} v^2 + \frac{m(m+1)(m+2)}{1.2.3} v^3 + \&c.;$$

and by means of this formula, the expansion of $(a+x)^m$ may be made to assume various different forms.

Thus, since $\dfrac{a+x}{a} = \dfrac{1}{1 - \dfrac{x}{a+x}}$, we shall have

$$(a+x)^m = \frac{a^m}{\left(1 - \dfrac{x}{a+x}\right)^m} = a^m \left(1 - \frac{x}{a+x}\right)^{-m}$$

$$= a^m \left\{1 + m\left(\frac{x}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{x}{a+x}\right)^2\right.$$

$$\left. + \frac{m(m+1)(m+2)}{1.2.3}\left(\frac{x}{a+x}\right)^3 + \&c.\right\}.$$

Similarly, since $\dfrac{a+x}{x} = \dfrac{1}{1 - \dfrac{a}{a+x}}$, we shall have

$$(a+x)^m = x^m \left\{1 + m\left(\frac{a}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a}{a+x}\right)^2 + \&c.\right\}.$$

Again, because $\dfrac{a+x}{2x} = \dfrac{1}{1 - \dfrac{a-x}{a+x}}$, we readily obtain

$$(a+x)^m = (2x)^m \left(1 - \frac{a-x}{a+x}\right)^{-m}$$

$$= 2^m x^m \left\{1 + m\left(\frac{a-x}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a-x}{a+x}\right)^2 + \&c.\right\}.$$

Similarly, from $\dfrac{a+x}{2a} = \dfrac{1}{1 + \dfrac{a-x}{a+x}}$, is deduced

$$(a+x)^m = 2^m a^m \left\{1 - m\left(\frac{a-x}{a+x}\right) + \frac{m(m+1)}{1.2}\left(\frac{a-x}{a+x}\right)^2 - \&c.\right\}:$$

and the greatest terms, or those after which the series begin to converge, may be determined as in article (197).

205. The use of the Theorem investigated in the preceding pages will now be evinced by its application to a few examples.

Ex. 1. Required the fifth power of $2x \pm 3y$.

Here $(2x \pm 3y)^5 = (2x)^5 \left\{1 \pm \dfrac{3y}{2x}\right\}^5$, and by substituting in the formula of the expansion of $(1+v)^m$, the quantities $\pm \dfrac{3y}{2x}$ and 5, in the places of $v$ and $m$ respectively, we get

$$(2x \pm 3y)^5 = 32x^5 \left\{1 \pm 5\left(\frac{3y}{2x}\right) + \frac{5(5-1)}{1.2}\left(\frac{3y}{2x}\right)^2\right.$$

$$\pm \frac{5(5-1)(5-2)}{1.2.3}\left(\frac{3y}{2x}\right)^3 + \frac{5(5-1)(5-2)(5-3)}{1.2.3.4}\left(\frac{3y}{2x}\right)^4$$

$$\left.\pm \frac{5(5-1)(5-2)(5-3)(5-4)}{1.2.3.4.5}\left(\frac{3y}{2x}\right)^5\right\}$$

$$=32x^5\left\{1 \pm 5\left(\frac{3y}{2x}\right) + 10\left(\frac{3y}{2x}\right)^2 \pm 10\left(\frac{3y}{2x}\right)^3 + 5\left(\frac{3y}{2x}\right)^4 \pm \left(\frac{3y}{2x}\right)^5\right\}$$

$$=32x^5 \pm 240x^4y + 720x^3y^2 \pm 1080x^2y^3 + 810xy^4 \pm 243y^5.$$

Ex. 2. Required the square root of $a+x$ in an infinite series.

Here $\sqrt{a+x} = (a+x)^{\frac{1}{2}} = a^{\frac{1}{2}}\left(1+\frac{x}{a}\right)^{\frac{1}{2}}$, and putting $\frac{x}{a}$ and $\frac{1}{2}$ in the places of $v$ and $m$ as before, we shall, after performing the requisite reductions, have

$$\sqrt{a+x}$$

$$=a^{\frac{1}{2}} + \frac{1}{2a^{\frac{1}{2}}}\,\frac{x}{1} - \frac{1.1}{2^2a^{\frac{3}{2}}}\,\frac{x^2}{1.2} + \frac{1.1.3}{2^3a^{\frac{5}{2}}}\,\frac{x^3}{1.2.3} - \frac{1.1.3.5}{2^4a^{\frac{7}{2}}}\,\frac{x^4}{1.2.3.4} + \&c.$$

$$=a^{\frac{1}{2}} + \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} + \frac{x^3}{16a^{\frac{5}{2}}} - \frac{5x^4}{128a^{\frac{7}{2}}} + \&c.$$

Ex. 3. Let it be required to convert $\frac{a^2}{(1-x^2)^{\frac{1}{5}}}$ in a series.

In this case, $\frac{1}{(1-x^2)^{\frac{1}{5}}} = (1-x^2)^{-\frac{1}{5}}$, so that if for $v$ and $m$ there be substituted $-x^2$ and $-\frac{1}{5}$ respectively, we shall have, after reduction,

$$\frac{a^2}{(1-x^2)^{\frac{1}{5}}} = a^2 + \frac{a^2x^2}{5} + \frac{6a^2x^4}{5.10} + \frac{6.11a^2x^6}{5.10.15} + \frac{6.11.16a^2x^8}{5.10.15.20} + \&c.$$

Ex. 4. If we assume $(a+\sqrt{b})^{\frac{1}{m}} = x+\sqrt{y}$, then will

$$a+\sqrt{b}=(x+\sqrt{y})^m$$

$$=x^m+mx^{m-1}\sqrt{y}+\frac{m(m-1)}{1.2}x^{m-2}y+\&c.;$$

whence equating the rational and surd quantities on both sides respectively, we shall have

$$a=x^m+\frac{m(m-1)}{1.2}x^{m-2}y+\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}x^{m-4}y^2+\&c.:$$

$$\sqrt{b}=mx^{m-1}\sqrt{y}+\frac{m(m-1)(m-2)}{1.2.3}x^{m-3}y\sqrt{y}+\&c.:$$

$$\therefore\ a-\sqrt{b}=x^m-mx^{m-1}\sqrt{y}+\frac{m(m-1)}{1.2}x^{m-2}y-\&c.$$

which is manifestly $=(x-\sqrt{y})^m$,

and $\therefore\ (a-\sqrt{b})^{\frac{1}{m}}=x-\sqrt{y}$.

Here it is understood that $\sqrt{b}$ and $\sqrt{y}$ involve the same irrational factor; and in the same manner, if $\sqrt{a}$ and $\sqrt{b}$ involve the same irrational factors as $\sqrt{x}$ and $\sqrt{y}$ respectively, and $(\sqrt{a}+\sqrt{b})^{\frac{1}{m}}=\sqrt{x}+\sqrt{y}$, where $m$ is an odd number, then will $(\sqrt{a}-\sqrt{b})^{\frac{1}{m}}=\sqrt{x}-\sqrt{y}$.

Ex. 5. By the application of the general formula, we have

$$(a+b\sqrt{-1})^m$$

$$=a^m+ma^{m-1}(b\sqrt{-1})+\frac{m(m-1)}{1.2}a^{m-2}(b\sqrt{-1})^2+\&c.$$

$$= a^m + m a^{m-1} b \sqrt{-1} - \frac{m(m-1)}{1 \cdot 2} a^{m-2} b^2 - \&c.$$

$$= a^m - \frac{m(m-1)}{1 \cdot 2} a^{m-2} b^2 + \&c.$$

$$+ \left( m a^{m-1} b - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^3 + \&c. \right) \sqrt{-1} :$$

similarly, we shall have $(a - b\sqrt{-1})^m$

$$= a^m - \frac{m(m-1)}{1 \cdot 2} a^{m-2} b^2 + \&c.$$

$$- \left( m a^{m-1} b - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^3 + \&c. \right) \sqrt{-1} :$$

whence, by addition and subtraction, we obtain

$$(a + b\sqrt{-1})^m + (a - b\sqrt{-1})^m$$

$$= 2 \left\{ a^m - \frac{m(m-1)}{1 \cdot 2} a^{m-2} b^2 + \&c. \right\},$$ which is possible:

and $(a + b\sqrt{-1})^m - (a - b\sqrt{-1})^m$

$$= 2 \left\{ m a^{m-1} b - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^3 + \&c. \right\} \sqrt{-1},$$

which is impossible.

206. The Binomial Theorem may be advantageously employed in extracting the roots of numerical magnitudes where an approximation only is necessary, and the ordinary process would be exceedingly tedious.

Ex. 1. To extract the square root of 10 in a series.

The square root of $10 = \sqrt{9+1} = 3\sqrt{1+\frac{1}{9}}$

$$= 3\left\{1 + \frac{1}{2}\left(\frac{1}{9}\right) - \frac{1}{2.4}\left(\frac{1}{9}\right)^2 + \frac{1}{2.8}\left(\frac{1}{9}\right)^3 - \text{\&c.}\right\}$$

$$= 3 + \frac{1}{2.3} - \frac{1}{2.4.27} + \frac{1}{2.8.243} - \text{\&c.}$$

Ex. 2. Required the fifth root of 260.

Here $260 = 243 + 17 = 3^5 + 17$: and therefore if $a = 3^5$ and $x = 17$, we shall have

$$\sqrt[5]{260} = (a+x)^{\frac{1}{5}}$$

$$= a^{\frac{1}{5}}\left\{1 + \frac{1}{5}\left(\frac{x}{a}\right) - \frac{1.4}{2.5^2}\left(\frac{x}{a}\right)^2 + \frac{1.4.9}{2.3.5^3}\left(\frac{x}{a}\right)^3 - \text{\&c.}\right\};$$

now, of the quantities between the brackets,

the first term $= 1 = A$;

the second term $= \dfrac{Ax}{5a} = .0139918 = B$;

the third term $= \dfrac{2Bx}{5a} = .0003915 = C$;

the fourth term $= \dfrac{3Cx}{5a} = .0000164 = D$;

&c. ......................................

$\therefore$ by substituting these values in the expression above, we have

$$\sqrt[5]{260} = 3\,\{1.0136159 \text{ \&c.}\} = 3.0408477 \text{ \&c.}$$

Ex. 3. Let it be required to find the power expressed by $\frac{9}{2}$, of the fraction $\frac{11}{6}$.

Here we have $\left(\frac{11}{6}\right)^{\frac{9}{2}} = \left(1 + \frac{5}{6}\right)^{\frac{9}{2}}$

$$= 1 + \frac{9}{2}\left(\frac{5}{6}\right) + \frac{9 \,.\, 7}{2^3}\left(\frac{5}{6}\right)^2 + \frac{9 \,.\, 7 \,.\, 5}{2^4 \,.\, 3}\left(\frac{5}{6}\right)^3 + \frac{9 \,.\, 7 \,.\, 5 \,.\, 3}{2^5 \,.\, 3 \,.\, 4}\left(\frac{5}{6}\right)^4 + \&c.:$$

wherein by (197) the place of the greatest term is denoted by the whole number next greater than $(m+1)\dfrac{v}{v+1}$, which in this instance

$$= \frac{11}{2}\left\{\frac{\frac{5}{6}}{\frac{5}{6}+1}\right\} = 2\tfrac{1}{2}:$$

$\therefore$ the third term is here the greatest, and it is readily seen that the fourth term which is $\dfrac{4375}{1152}$, is less than the third term which is $\dfrac{175}{32}$: in other words, the series begins to converge after the third term.

If it were proposed to find the expansion of the quantity

$$\frac{1}{\left(\frac{11}{6}\right)^{\frac{9}{2}}}, \text{ or of } \left(1 + \frac{5}{6}\right)^{-\frac{9}{2}},$$

the place of the greatest term would by (198) be expressed by the whole number next greater than

$$(m+1)\frac{v}{v-1} = -\frac{7}{2}\left\{\frac{\frac{5}{6}}{\frac{5}{6}-1}\right\} = 17\tfrac{1}{2};$$

or the 18th term is the greatest, and may be found immediately by means of (188).

So, likewise in the developement of the fraction

$$\frac{1}{\left(\frac{1}{6}\right)^{\frac{9}{2}}} = \left(1 - \frac{5}{6}\right)^{-\frac{9}{2}},$$

the determining quantity being $17\frac{1}{2}$, the 18th term will be the greatest as above.

In order that the terms of the series may decrease with sufficient rapidity, it will sometimes be necessary so to transform the surd, that the latter term of the quantity affected by the index may be a small proper fraction.

207. Without, however, exhibiting the required roots in series, an useful practical approximation to the higher roots of numerical quantities may be easily deduced from the same theorem.

Let $a$ be an approximate value of the $m^{\text{th}}$ root of $N$, such that $\sqrt[m]{N} = a + x$, $x$ being a very small quantity:

$$\therefore\ N = (a+x)^m = a^m + m a^{m-1} x + \frac{m(m-1)}{1 \cdot 2} a^{m-2} x^2 + \text{\&c.}$$

$$\text{whence } N - a^m = m a^{m-1} x \text{ nearly, and } \therefore\ x = \frac{N - a^m}{m a^{m-1}}:$$

$$\text{but } N - a^m = \left(m a^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^{m-2} x\right) x, \text{ nearly,}$$

$$= \left\{m a^{m-1} + \frac{(m-1)(N - a^m)}{2a}\right\} x, \text{ nearly:}$$

$$\therefore\ x = \frac{2a(N - a^m)}{(m+1)a^m + (m-1)N},$$

$$\text{and } \therefore\ \sqrt[m]{N} = a + x$$

$$=a+\frac{2a(N-a^m)}{(m+1)a^m+(m-1)N}$$

$$=\frac{(m+1)N+(m-1)a^m}{(m-1)N+(m+1)a^m}\times a, \text{ nearly,}$$

which is a nearer approximation to the true root: let this be called $a'$, and by a repetition of the same process, we shall find

$$\sqrt[m]{N}=\frac{(m+1)N+(m-1)a'^m}{(m-1)N+(m+1)a'^m}\times a', \text{ nearly,}$$

which is still nearer to the true value: and so on, to any required degree of exactness.

208. Cor. If $N=a^m\pm b$, we shall have by substitution

$$\sqrt[m]{N}=a\pm\frac{2ab}{(m+1)a^m+(m-1)(a^m\pm b)}$$

$$=a\pm\frac{2ab}{2ma^m\pm(m-1)b},$$

as a first approximation: and if this approximate value be called $a'$ and we assume again $N=a'^m\pm b'$, then will

$$\sqrt[m]{N}=a'\pm\frac{2a'b'}{2ma'^m\pm(m-1)b'},$$

which is a second and nearer approximation, and so on.

Ex. If $m=2$, then will either of the quantities

$$\left\{\frac{3N+a^2}{N+3a^2}\right\}a \text{ and } a\pm\frac{2ab}{4a^2\pm b},$$

be a first approximation to the square root of $N$ or $a^2\pm b$:

if $m=3$, we shall similarly have

$$\left\{\frac{4N+2a^2}{2N+4a^2}\right\}a \text{ or } a\pm\frac{2ab}{6a^3\pm 2b},$$

for a first approximation to the cube root of $N$ or $a^3 \pm b$: and so on.

209. *To extract, when possible, the* m<sup>th</sup> *root of a binomial, one or both of whose terms are possible quadratic surds.*

Let the proposed quadratic surd be $\sqrt{a}+\sqrt{b}$ wherein $\sqrt{a}$ is greater than $\sqrt{b}$, and let $n$, $u$ and $v$ be such that

$$\sqrt[m]{(\sqrt{a}+\sqrt{b})\times\sqrt{n}}=\sqrt{u}+\sqrt{v};$$

$$\therefore \sqrt[m]{(\sqrt{a}-\sqrt{b})\times\sqrt{n}}=\sqrt{u}-\sqrt{v}, \text{ by Ex. 4. of (205):}$$

whence by multiplication we have $\sqrt[m]{(a-b)\times n}=u-v$: let now $n$ be assumed of such a magnitude that

$$(a-b)\,n=p^m,$$

whence we shall have $u-v=p$:

$$\text{again, } \sqrt[m]{(\sqrt{a}+\sqrt{b})^2\times n}+\sqrt[m]{(\sqrt{a}-\sqrt{b})^2\times n}$$
$$=(\sqrt{u}+\sqrt{v})^2+(\sqrt{u}-\sqrt{v})^2$$
$$=2\,(u+v),$$

which is obviously an integral quantity: wherefore if $q$ and $r$ be approximate values of

$$\sqrt[m]{(\sqrt{a}+\sqrt{b})^2\times n} \text{ and } \sqrt[m]{(\sqrt{a}-\sqrt{b})^2\times n},$$

such that one of them is greater, and the other less than the true value, we shall have

$$u+v=\frac{q+r}{2};$$

$$\text{also, } u-v=p,$$

and from these we find $u = \dfrac{q+r+2p}{4}$,

$$v = \frac{q+r-2p}{4}:$$

$$\text{and } \therefore \sqrt[m]{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{u}+\sqrt{v}}{\sqrt[2m]{n}}$$

$$= \frac{1}{2\sqrt[2m]{n}}\{\sqrt{q+r+2p}+\sqrt{q+r-2p}\},$$

whenever the root can be so exhibited.

Similarly, from what has been proved above, we get

$$\sqrt[m]{\sqrt{a}-\sqrt{b}} = \frac{1}{2\sqrt[2m]{n}}\left\{\sqrt{q+r+2p}-\sqrt{q+r-2p}\right\}.$$

Ex. 1. Let it be required to extract the cube root of $9+4\sqrt{5}$.

Here $a-b=81-80=1$, and $\therefore$ $n=1$ and $u-v=1$:

$$\text{again, } \sqrt[3]{(\sqrt{a}+\sqrt{b})^2\times n} = 6+\delta,$$

$$\text{and } \sqrt[3]{(\sqrt{a}-\sqrt{b})^2\times n} = 1-\delta;$$

$$\text{whence we obtain } u+v = \frac{6+\delta+1-\delta}{2} = \frac{7}{2}:$$

and from the two equations $u+v=\frac{7}{2}$ and $u-v=1$, are readily deduced $u=\frac{9}{4}$ and $v=\frac{5}{4}$;

$$\text{and } \therefore \sqrt[3]{9+4\sqrt{5}} = \frac{3+\sqrt{5}}{2},$$

which equation may easily be verified.

Ex. 2. To extract the fifth root of $41 + 29\sqrt{2}$, we have $a - b = (29\sqrt{2})^2 - (41)^2 = 1$, whence $n = 1$ and $\therefore\ u - v = 1$:

$$\text{also, } \sqrt[5]{(29\sqrt{2} + 41)^2} = \sqrt[5]{3363 + 2378\sqrt{2}} = 5 + \delta,$$

$$\text{and } \sqrt[5]{(29\sqrt{2} - 41)^2} = \sqrt[5]{3363 - 2378\sqrt{2}} = 1 - \delta:$$

wherefore $2(u + v) = 5 + \delta + 1 - \delta = 6$ and $u + v = 3$:

$\therefore$ from the two equations $u + v = 3$ and $u - v = 1$, we have immediately $\sqrt{u} = \sqrt{2}$ and $\sqrt{v} = 1$, so that

$$\sqrt[5]{41 + 29\sqrt{2}} = \sqrt{2} + 1, \text{ the required root.}$$

210. The expansions of trinomials, quadrinomials, &c. may also be obtained by means of the Binomial Theorem, by considering two or more of their terms as one: thus

$$(a + b + c)^m = \{a + (b + c)\}^m = a^m + m a^{m-1}(b + c)$$
$$+ \frac{m(m-1)}{1 \cdot 2} a^{m-2}(b + c)^2 + \&c.;$$

$$(a + b + c + d)^m = \{(a + b) + (c + d)\}^m = (a + b)^m$$
$$+ m(a + b)^{m-1}(c + d) + \frac{m(m-1)}{1 \cdot 2}(a + b)^{m-2}(c + d)^2 + \&c.;$$

and so on, in each of which the developements indicated must be effected, and the terms collected and arranged according to the dimensions of one of the letters involved: this will, of course, be very tedious and it may be superseded by the following theorem.

## III. The Multinomial Theorem.

211. The *Multinomial Theorem* is a formula for expanding or developing any power or root of an Algebraical quantity consisting of more than two terms.

In order to investigate this theorem, we shall premise the following proposition.

212. *In the expansion of* $(a+b+c+\&c.)^m$, *to find the coefficient of the literal product* $a^\alpha\, b^\beta\, c^\gamma$ &c.

For $b+c+d+$ &c. put $b'$, and let $m=\alpha+\beta'$; then in the expansion of $(a+b')^m$, one of the terms will be

$$\frac{m\,(m-1)\,(m-2)\,.\,\&c.\,(m-\beta'+1)}{1\,.\,2\,.\,3\,.\,\&c.\,\beta'}\,a^{m-\beta'}\,b'^{\beta'}$$

$$=\frac{1\,.\,2\,.\,3\,.\,\&c.\,m}{(1\,.\,2\,.\,3\,.\,\&c.\,\alpha)\,(1\,.\,2\,.\,3\,.\,\&c.\,\beta')}\,a^\alpha\,b'^{\beta'},$$

by multiplying both the numerator and denominator by $1\,.\,2\,.\,3\,.\,\&c.\,\alpha$;

again, for $b'$ put $b+c'$ and let $\beta'=\beta+\gamma'$; then in the expansion of $(b+c')^{\beta'}$, it follows as before that one of the terms will be

$$\frac{1\,.\,2\,.\,3\,.\,\&c.\,\beta'}{(1\,.\,2\,.\,3\,.\,\&c.\,\beta)\,(1\,.\,2\,.\,3\,.\,\&c.\,\gamma')}\,b^\beta\,c'^{\gamma'},$$

wherein $\alpha+\beta+\gamma'=m$; and by combining this with the former, we have one of the terms of the expansion of $(a+b+c')^m$

$$=\frac{1\,.\,2\,.\,3\,.\,\&c.\,m}{(1\,.\,2\,.\,3\,.\,\&c.\,\alpha)\,(1\,.\,2\,.\,3\,.\,\&c.\,\beta)\,(1\,.\,2\,.\,3\,.\,\&c.\,\gamma')}\,a^\alpha\,b^\beta\,c'^{\gamma'}:$$

and proceeding as above and substituting $c+d'$ for $c'$ and $\gamma+\delta'$ for $\gamma'$, &c. we shall obtain the general term of $(a+b+c+\&c.)^m$

$$=\frac{1.2.3\,.\,\&c.\;m}{(1.2.3\,.\,\&c.\;\alpha)\,(1.2.3\,.\,\&c.\;\beta)\,(1.2.3\,.\,\&c.\;\gamma)\;\&c.}\,a^\alpha b^\beta c^\gamma\;\&c.$$

wherein the values of $\alpha$, $\beta$, $\gamma$, &c. are obviously subject to the condition that $\alpha+\beta+\gamma+$ &c. $=m$.

213. Cor. 1. By assigning to each of the quantities $a$, $\beta$, $\gamma$, &c. in its turn all possible integral values which the equation of condition, $a + \beta + \gamma + \&c. = m$, admits of, all the terms of the expansion of $(a + b + c + \&c.)^m$ may be obtained.

214. Cor. 2. If $\beta + \gamma + \delta + \&c. = \phi$, then we shall have $a = m - \phi$, and the general term becomes

$$\frac{m(m-1)(m-2).\&c.(m-\phi+1)}{(1.2.3.\&c.\beta)(1.2.3.\&c.\gamma)(1.2.3.\&c.\delta)\&c.} a^{m-\phi} b^{\beta} c^{\gamma} d^{\delta} \&c.,$$

by expunging from the numerator and denominator the common factor $1.2.3.\&c.a$ or $1.2.3.\&c.(m-\phi)$.

215. *To find the terms of the expansion of*

$$(a + bx + cx^2 + \&c. + lx^p)^m;$$

*or which is the same thing, to investigate the Multinomial Theorem.*

We have already demonstrated that the general term of the expansion of $(a + b + c + \&c.)^m$ is

$$\frac{1.2.3.\&c.m}{(1.2.3.\&c.a)(1.2.3.\&c.\beta)(1.2.3.\&c.\gamma)\&c.} a^{a} b^{\beta} c^{\gamma} \&c.,$$

subject to the condition that $a + \beta + \gamma + \&c. = m$:

wherefore if we substitute $bx$, $cx^2$, &c., in the places of $b$, $c$, &c. we shall have the general term of the expansion of

$$(a + bx + cx^2 + \&c. + lx^p)^m$$

$$= \frac{1.2.3.\&c.m}{(1.2.3.\&c.a)(1.2.3.\&c.\beta)(1.2.3.\&c.\gamma)\&c.} a^{a} b^{\beta} x^{\beta} c^{\gamma} x^{2\gamma} \&c.$$

$$= \frac{1.2.3.\&c.m}{(1.2.3.\&c.a)(1.2.3.\&c.\beta)(1.2.3.\&c.\gamma)\&c.} a^{a} b^{\beta} c^{\gamma} \&c. x^{\beta+2\gamma+\&c.};$$

and all the terms, in which the index of $x$ is $\beta+2\gamma+$ &c., may be derived from this by giving to $a$, $\beta$, $\gamma$, &c., all the different positive integral values of which they are capable, consistently with the limitation that

$$a+\beta+\gamma+\&c.=m:$$

and if $\beta+2\gamma+\&c.$ be assumed $=n$,

the two equations of condition become

$$a+\beta+\gamma+\&c.=m$$

$$\text{and } \beta+2\gamma+3\delta+\&c.=n.$$

216. Cor. If $\beta+\gamma+\delta+\&c.=\phi$, the whole coefficient of $x^n$ will be obtained by forming all the possible rational literal products and corresponding coefficients, the conditions being that

$$\beta+\gamma+\delta+\&c.=\phi$$

$$\text{and } \beta+2\gamma+3\delta+\&c.=n.$$

Ex. Required the coefficient of $x^n$ in the expansion of $(1+x+x^2)^m$.

The general term in this instance is

$$\frac{1.2.3.\&c.\,m.}{(1.2.3.\&c.\,a)\,(1.2.3.\&c.\,\beta)\,(1.2.3.\&c.\,\gamma)}\,a^{\alpha}b^{\beta}c^{\gamma}x^{\beta+2\gamma}:$$

and here $a=b=c=1$:

also since $a+\beta+\gamma=m$ and $\beta+2\gamma=n$,

we shall easily find $a=m-n+\gamma$ and $\beta=n-2\gamma$,

so that it will now be sufficient to assign all possible positive integral values to $\gamma$ in the general term which in this case becomes

$$\frac{1.2.3.\&c.\,m}{\{1.2.3.\&c.\,(m-n+\gamma)\}\,\{1.2.3.\&c.\,(n-2\gamma)\}\,\{1.2.3.\&c.\,\gamma\}}\,x^n:$$

and from this we may obtain each term of the coefficients of $x^n$ in succession, by giving to $\gamma$ all possible positive integral values beginning with 0.

Let $A$, $B$, $C$, $D$, &c. represent the successive terms of the coefficient of $x^n$: then

if $\gamma = 0$,

$$\text{we have } A = \frac{1.2.3.\&\text{c. } m}{\{1.2.3.\&\text{c. } (m-n)\}\{1.2.3.\&\text{c. } n\}}$$

$$= \frac{m(m-1)(m-2).\&\text{c.}(m-n+1)}{1.2.3.\&\text{c. } n}:$$

if $\gamma = 1$,

$$\text{then } B = \frac{1.2.3.\&\text{c. } m}{\{1.2.3.\&\text{c. } (m-n+1)\}\{1.2.3.\&\text{c. } (n-2)\}\{1\}}$$

$$= \frac{m(m-1)(m-2).\&\text{c. }(m-n+2)(m-n+1)(m-n).\&\text{c. } 3.2.1}{\{1.2.3.\&\text{c. } (m-n+1)\}\{1.2.3.\&\text{c. } (n-2)\}\{1\}}$$

$$= \frac{m(m-1)(m-2).\&\text{c.}(m-n+2)}{\{1.2.3.\&\text{c. } (n-2)\}\{1\}}:$$

if $\gamma = 2$, then

$$C = \frac{1.2.3.\&\text{c. } m}{\{1.2.3.\&\text{c. } (m-n+2)\}\{1.2.3.\&\text{c. } (n-4)\}\{1.2\}}$$

$$= \frac{m(m-1)(m-2).\&\text{c.}(m-n+3)}{\{1.2.3.\&\text{c. } (n-4)\}\{1.2\}},$$

by reduction as above:

if $\gamma = 3$,

$$\text{then } D = \frac{m(m-1)(m-2) \text{ \&c. } (m-n+4)}{\{1.2.3. \text{ \&c. } (n-6)\}\{1.2.3\}},$$

by a similar process: and so of succeeding numbers; and hence the law of the formation of the rest of the terms is manifest, and thus the coefficient required is obtained.

217. Ex. If $n$ be made equal to the numbers 1, 2, 3, 4, &c. in order, the successive terms of the expansion of $(1+x+x^2)^m$, beginning with the first term that involves $x$ will be obtained.

Thus, if $A_1$, $B_1$, $C_1$, &c., $A_2$, $B_2$, $C_2$, &c., and so on, denote the corresponding values of $A$, $B$, $C$, &c., we shall have

$$\text{if } n=1, \quad A_1 = m, \quad B_1 = 0, \quad \text{\&c.};$$

$$\text{if } n=2, \quad A_2 = \frac{m(m-1)}{1.2}, \quad B_2 = m, \quad C_2 = 0, \quad \text{\&c.};$$

$$\text{if } n=3, \quad A_3 = \frac{m(m-1)(m-2)}{1.2.3}, \quad B_3 = \frac{m(m-1)}{1.2}, \quad C_3 = 0, \quad \text{\&c.};$$

$$\text{if } n=4, \quad A_4 = \frac{m(m-1)(m-2)(m-3)}{1.2.3.4},$$

$$B_4 = \frac{m(m-1)(m-2)}{1.2}, \quad C_4 = \frac{m(m-1)}{1.2}, \quad D_4 = 0, \quad \text{\&c.};$$

$$\text{if } n=5, \quad A_5 = \frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5},$$

$$B_5 = \frac{m(m-1)(m-2)(m-3)}{1.2.3}, \quad C_5 = \frac{m(m-1)(m-2)}{1.2}, \quad D_5 = 0,$$

&c.;

and so on, for succeeding values of $n$:

$\therefore$ the expansion of $(1+x+x^2)^m$, which is represented by

$$1+(A_1+B_1+C_1+\&c.)\,x+(A_2+B_2+C_2+\&c.)\,x^2$$
$$+(A_3+B_3+C_3+\&c.)\,x^3+(A_4+B_4+C_4+\&c.)\,x^4$$
$$+(A_5+B_5+C_5+\&c.)\,x^5+\&c.$$

will be

$$1+m\left|\,x+\begin{array}{l}\dfrac{m(m-1)}{1.2}\\[2ex] +\;m\end{array}\right|\,x^2+\begin{array}{l}\dfrac{m(m-1)(m-2)}{1.2.3}\\[2ex] +\;\dfrac{m(m-1)}{1.2}\end{array}\Bigg|\,x^3$$

$$+\begin{array}{l}\dfrac{m(m-1)(m-2)(m-3)}{1.2.3}\\[2ex] +\;\dfrac{m(m-1)(m-2)}{1.2}\\[2ex] +\;\dfrac{m(m-1)}{1.2}\end{array}\Bigg|\,x^4$$

$$+\begin{array}{l}\dfrac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5}\\[2ex] +\;\dfrac{m(m-1)(m-2)(m-3)}{1.2.3}\\[2ex] +\;\dfrac{m(m-1)(m-2)}{1.2}\end{array}\Bigg|\,x^5+\&c.$$

and the law, according to which the succeeding terms will be formed, is manifest.

218. If $a_0+a_1x+a_2x^2+a_3x^3+\&c.+a_px^p$ be the proposed multinomial, the theorem for its expansion may, by the method of indeterminate coefficients, be exhibited in the following form :

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. + a_p x^p)^m$$

$$= a_0^m + \begin{array}{l|l} m a_1 t_0 & \dfrac{x}{a_0} \end{array} + \begin{array}{l|l} (2m-0)\,a_2 t_0 & \dfrac{x^2}{2a_0} \\ +\,(1m-1)\,a_1 t_1 & \end{array} + \begin{array}{l|l} (3m-0)\,a_3 t_0 & \dfrac{x^3}{3a_0} \\ +\,(2m-1)\,a_2 t_1 & \\ +\,(1m-2)\,a_1 t_2 & \end{array}$$

$$+ \begin{array}{l|l} (4m-0)\,a_4 t_0 & \dfrac{x^4}{4a_0} \\ +\,(3m-1)\,a_3 t_1 & \\ +\,(2m-2)\,a_2 t_2 & \\ +\,(1m-3)\,a_1 t_3 & \end{array} + \&c.,$$

where $t_0$, $t_1$, $t_2$, $t_3$, &c. are the terms of the expansion in which the indices of $x$ are 0, 1, 2, 3, &c.

## IV. The Exponential Theorem.

219. If in the expansion of $(1+v)^m$ the terms be arranged according to the powers of the exponent $m$, instead of those of $v$ as was done in the Binomial Theorem, the formula thence arising is called the *Exponential Theorem.*

220. Since by (185) we have seen that

$$(1+v)^m = 1 + mv + \frac{m(m-1)}{1.2}v^2 + \frac{m(m-1)(m-2)}{1.2.3}v^3 + \&c.;$$

if we suppose $v = a - 1$ and $m = x$, we shall obviously have

$$a^x = 1 + \left\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.\right\}x$$
$$+ Bx^2 + Cx^3 + Dx^4 + \&c. \qquad (a)$$

where $B$, $C$, $D$, &c. consist of $a-1$ and its powers, and are therefore independent of $x$:

let therefore the expression $(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-$ &c. be assumed $=A$:

$$\text{so that } a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$$

$$\begin{aligned}
\therefore\ a^{2x} = a^x \times a^x = 1 &+ Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \\
&+ Ax + A^2x^2 + ABx^3 + ACx^4 + \&c. \\
&+ Bx^2 + ABx^3 + B^2x^4 + \&c. \\
&+ Cx^3 + ACx^4 + \&c. \\
&+ Dx^4 + \&c. \\
&+ \&c.
\end{aligned}$$

$$= 1 + 2Ax + (2B + A^2)x^2 + (2C + 2AB)x^3 + (2D + 2AC + B^2)x^4 + \&c.:$$

but by $(a)$ we shall have also

$$\begin{aligned}
a^{2x} &= 1 + A(2x) + B(2x)^2 + C(2x)^3 + D(2x)^4 + \&c. \\
&= 1 + 2Ax + 2^2Bx^2 + 2^3Cx^3 + 2^4Dx^4 + \&c.:
\end{aligned}$$

whence equating the coefficients of the same powers of $x$ in these two expressions for $a^{2x}$, we obtain

$$2^2B = 2B + A^2, \qquad \therefore\ B = \frac{A^2}{1.2};$$

$$2^3C = 2C + 2AB, \qquad \therefore\ C = \frac{A^3}{1.2.3};$$

$$2^4D = 2D + 2AC + B^2, \qquad \therefore\ D = \frac{A^4}{1.2.3.4}; \text{ and so on:}$$

$$\text{wherefore } a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \&c.:$$

and thus the law of the coefficients of the powers of $x$ is discovered, the quantity $A$ being equivalent to the series $(a-1)$ $-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-$ &c. continued to $x$ terms, if $x$ be a positive integral quantity, and to infinity, if it be fractional or negative.

It may here be observed, that in the latter cases this is merely an analytical value of $a^x$, the sign $=$ being used in the sense explained in (88).

221. Cor. 1. By substituting in both sides of the equation above deduced $-x$ in the place of $x$, we shall have

$$a^{-x} = 1 - Ax + \frac{A^2 x^2}{1.2} - \frac{A^3 x^3}{1.2.3} + \frac{A^4 x^4}{1.2.3.4} - \&c.;$$

and therefore by addition and subtraction, there result

$$a^x + a^{-x} = 2\left\{1 + \frac{A^2 x^2}{1.2} + \frac{A^4 x^4}{1.2.3.4} + \&c.\right\},$$

$$\text{and } a^x - a^{-x} = 2Ax\left\{1 + \frac{A^2 x^2}{1.2.3} + \frac{A^4 x^4}{1.2.3.4.5} + \&c.\right\}.$$

222. Cor. 2. If in the place of $x$ we put $mx$, then since the value of $A$ is not altered, we have

$$a^{mx} = 1 + mAx + \frac{m^2 A^2 x^2}{1.2} + \frac{m^3 A^3 x^3}{1.2.3} + \&c.$$

$$\text{and } \therefore \left(1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} + \&c.\right)^m =$$

$$1 + mAx + \frac{m^2 A^2 x^2}{1.2} + \frac{m^3 A^3 x^3}{1.2.3} + \&c.$$

223. Cor. 3. If $B = (b-1) - \frac{1}{2}(b-1)^2 + \frac{1}{3}(b-1)^3 - \&c.$, we shall in the same manner have

$$b^x = 1 + Bx + \frac{B^2 x^2}{1.2} + \frac{B^3 x^3}{1.2.3} + \&c.$$

and continuing the same kind of notation, we obtain

$$c^x = 1 + Cx + \frac{C^2 x^2}{1.2} + \frac{C^3 x^3}{1.2.3} + \&c. \text{ and so on;}$$

whence we immediately obtain

$$\left\{1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\&c.\right\}\left\{1+Bx+\frac{B^2x^2}{1.2}+\frac{B^3x^3}{1.2.3}+\&c.\right\}$$

$$\left\{1+Cx+\frac{C^2x^2}{1.2}+\frac{C^3x^3}{1.2.3}+\&c.\right\}\ \&c. = (abc\ \&c.)^x$$

$$=1+Px+\frac{P^2x^2}{1.2}+\frac{P^3x^3}{1.2.3}+\&c.$$

where the quantity denoted by $P$ is equivalent to

$$(abc\ \&c.-1)-\tfrac{1}{2}(abc\ \&c.-1)^2+\tfrac{1}{3}(abc\ \&c.-1)^3-\&c.$$

and if $a=b=c=\&c.$, the number of factors being $m$, we shall have

$$\left\{1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\&c.\right\}^m$$

$$=1+Px+\frac{P^2x^2}{1.2}+\frac{P^3x^3}{1.2.3}+\&c.$$

the quantity denoted by $P$ being then equivalent to

$$(a^m-1)-\tfrac{1}{2}(a^m-1)^2+\tfrac{1}{3}(a^m-1)^3-\&c.$$

Hence, combining this corollary with the preceding one, there will be obtained the following analytical result:

$$m\left\{(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.\right\}$$

$$=(a^m-1)-\tfrac{1}{2}(a^m-1)^2+\tfrac{1}{3}(a^m-1)^3-\&c.$$

224. Cor. 4. If we suppose $A=1$, the corresponding value of $a$ may be found.

For let $e$ be the required value of $a$ which will render $A=1$, $\therefore$ we shall obviously have

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c.;$$

from which, if $x$ be assumed $=1$, there immediately results

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$$

$=2.71828$ &c. by adding together the first ten terms:

also, if in $a^x = 1 + Ax + \dfrac{A^2x^2}{1.2} + \dfrac{A^3x^3}{1.2.3} + \&c.,$

we put $\dfrac{1}{A}$ in the place of $x$, we shall have

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. = e:$$

so that the connection subsisting between the quantities $a$ and $e$ is universally expressed by the equation

$$a^{\frac{1}{A}} = e \text{ or } a = e^A,$$

where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$ *in infinitum.*

Hence also from this article it appears that

$(e-1) - \frac{1}{2}(e-1)^2 + \frac{1}{3}(e-1)^3 - \&c.$ *in infinitum* $=1$.

225. By what has been proved, we shall be enabled to express the value of $\left(1 + \dfrac{x}{m}\right)^m$ when $m$ is indefinitely increased.

For, $\left(1 + \dfrac{x}{m}\right)^m = 1 + m\left(\dfrac{x}{m}\right) + \dfrac{m(m-1)}{1.2}\left(\dfrac{x}{m}\right)^2$

$$+ \frac{m(m-1)(m-2)}{1.2.3}\left(\frac{x}{m}\right)^3 + \&c.$$

$$= 1 + x + \frac{\left(1 - \frac{1}{m}\right)}{1 . 2} x^2 + \frac{\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)}{1 . 2 . 3} x^3 + \&c.$$

$$= 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \&c. \text{ when } m \text{ is indefinitely great,}$$

as appears from (89):

$$\text{therefore we have } \left(1 + \frac{x}{\infty}\right)^{\infty} = e^x,$$

which, when $x$ is assumed $= 1$, gives the numerical result

$$\left(1 + \frac{1}{\infty}\right)^{\infty} = 2.71828 \ \&c.$$

# CHAP. VIII.

## *On Ratio, Proportion and Variation.*

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### I. Ratio.

226. Def. *Ratio* is the relation which subsists between two quantities of the same kind with respect to magnitude, and is two-fold, *Arithmetical* and *Geometrical*.

Thus, if $a$ and $b$ be any two quantities whatever of the same kind, their arithmetical ratio will be obtained by determining how much the former exceeds the latter, and is denoted by $a-b$; and their geometrical ratio will be had by considering what multiple, part, or parts the former is of the latter, and is represented by $a:b$, which is equivalent to the fraction $\frac{a}{b}$.

The former of these views of the subject presents no propositions which are not obvious, and we shall on that account confine our attention to the latter.

In the algebraical expression $a:b$, the former quantity $a$ is called the *Antecedent*, and the latter $b$ the *Consequent* of the ratio: also, the ratio $a:b$ is said to be a ratio of *Equality* when $a=b$, and to be of *Greater* or *Less Inequality* according as $a$ is greater or less than $b$.

227. Cor. 1. Hence two ratios may be compared together: for the ratio $a:b$ is greater than, equal to, or less than the ratio $c:d$, according as the fraction $\frac{a}{b}$ is greater than, equal to, or less than the fraction $\frac{c}{d}$: that is, according as $\frac{ad}{bd}$

is greater than, equal to, or less than $\frac{bc}{bd}$, and therefore according as $ad$ is greater than, equal to, or less than $bc$.

228. Cor. 2. If the antecedents be equal, it is obvious that the ratio which has the less consequent is the greater; and if the consequents be equal, that the ratio is the greater which has the greater antecedent.

Ex. Which is the greater of the ratios; $a-x: a+x$ and $a^2-x^2: a^2+x^2$?

Here, according to article (226) the former ratio is expressed by $\frac{a-x}{a+x}$ and the latter by $\frac{a^2-x^2}{a^2+x^2}$:

$$\text{now } \frac{a-x}{a+x} = \frac{(a-x)(a^2+x^2)}{(a+x)(a^2+x^2)} = \frac{a^3-x^3-ax(a-x)}{(a+x)(a^2+x^2)};$$

$$\text{and } \frac{a^2-x^2}{a^2+x^2} = \frac{(a^2-x^2)(a+x)}{(a^2+x^2)(a+x)} = \frac{a^3-x^3+ax(a-x)}{(a+x)(a^2+x^2)};$$

and since the numerator of the latter fraction is greater than that of the former, it follows that the ratio, $a^2-x^2: a^2+x^2$, is greater than the ratio, $a-x: a+x$.

229. *A ratio of greater inequality is diminished, and of less inequality increased, by adding the same quantity to both its terms.*

Let $a : b$ be a ratio of inequality, and to each of its terms let the quantity $x$ be added, so that it becomes $a+x: b+x$; then the ratio $a : b$ is greater or less than the ratio $a+x: b+x$, according as $\frac{a}{b}$ is greater or less than $\frac{a+x}{b+x}$: that is, by (71), according as $\frac{ab+ax}{b(b+x)}$ is greater or less than $\frac{ab+bx}{b(b+x)}$:

First, suppose $a$ greater than $b$,

$$\text{then } \frac{ab+ax}{b(b+x)} \text{ is greater than } \frac{ab+bx}{b(b+x)},$$

and therefore

the ratio $a : b$ is greater than the ratio $a+x : b+x$:

that is, a ratio of greater inequality is diminished by adding the same quantity to both its terms.

Secondly, let $a$ be less than $b$;

$$\text{then will } \frac{ab+ax}{b(b+x)} \text{ manifestly be less than } \frac{ab+bx}{b(b+x)},$$

and therefore the ratio $a : b$ is less than the ratio $a+x : b+x$:

that is, a ratio of less inequality is increased by adding the same quantity to each term.

230. *A ratio of greater inequality is increased, and of less inequality diminished, by subtracting the same quantity from each term.*

First, let $a+x$ be greater than $b+x$, and therefore $a$ greater than $b$; then by the last article

$$a : b \text{ is greater than } a+x : b+x;$$

that is, a ratio of greater inequality is increased by subtracting the same quantity from each term.

Secondly, suppose $a+x$ less than $b+x$, and therefore $a$ less than $b$; then, as before,

$$a : b \text{ is less than } a+x : b+x;$$

or a ratio of less inequality is diminished by subtracting the same quantity from both its terms.

231. Def. If the antecedents of two or more ratios be multiplied together for a new antecedent, and their consequents together for a new consequent, the resulting ratio is said to be *compounded* of the others, and is sometimes termed their *Sum*: thus, if $a : b$, $c : d$, $e : f$, &c. be any ratios, the ratio arising from their composition, or their sum is $a\,c\,e$ &c. $: b\,d\,f$ &c.

232. Cor. 1. If $c = b$, $e = d$, &c., or the antecedent of the succeeding ratio be always the consequent of the preceding one, the compound ratio becomes $a\,b\,d$ &c. $x : b\,d\,f$ &c. $y$, which is equivalent to

$$\frac{a\,b\,d\,\&c.\,x}{b\,d\,f\,\&c.\,y} = \frac{a}{y} = a : y,$$

or, the ratio of the first antecedent to the last consequent.

233. Cor. 2. If the ratio $a : b$ be compounded with the ratio $x : y$, there results the ratio

$$ax : by \text{ or } \frac{ax}{by},$$

which is therefore greater or less than the ratio

$$a : b \text{ or } \frac{a}{b},$$

according as $x$ is greater or less than $y$:

likewise, if $x = y$, the ratio is not altered, in other words the ratio $a : b$ is equal to the ratio $ax : bx$, as appears also from (226).

234. Cor. 3. If each of the ratios $c : d$, $e : f$, &c. be equal to the ratio $a : b$, and there be $m$ such ratios, it is obvious that the compound ratio will be

$a\,a\,a$ &c. to $m$ factors $: b\,b\,b$ &c. to $m$ factors, or $a^m : b^m$:

and if $m$ be assumed equal to 1, 2, 3, &c. in succession, the

resulting ratio is styled the *simple, duplicate, triplicate,* &c. ratio of $a : b$, and sometimes its *single, double, treble,* &c.

By an extension of this kind of notation and nomenclature, the ratios $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, &c. are termed the *sub-duplicate, sub-triplicate,* &c. ratio of $a : b$, and in some cases, the *half, third,* &c. of it.

The ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the *sesquiplicate ratio* of $a : b$.

235. Cor. 4. Hence the indices,

$$2,\ 3,\ \&\text{c. } m;$$

$$\frac{1}{2},\ \frac{1}{3},\ \&\text{c.} \frac{1}{m}$$

have received the names of the *Measures of the Ratios*

$$a^2 : b^2,\ a^3 : b^3,\ \&\text{c. } a^m : b^m;$$

$$a^{\frac{1}{2}} : b^{\frac{1}{2}},\ a^{\frac{1}{3}} : b^{\frac{1}{3}},\ \&\text{c. } a^{\frac{1}{m}} : b^{\frac{1}{m}},$$

respectively.

236. If the difference between the antecedent and consequent of a ratio be small compared to either of them, useful practical approximations to the ratios just alluded to, may be readily obtained.

Thus, if $a + x : a$ be a proposed ratio wherein $x$ is very small compared to $a$, we shall have $(a + x)^m : a^m$

$$= \left(\frac{a+x}{a}\right)^m = \left(1 + \frac{x}{a}\right)^m = 1 + m\left(\frac{x}{a}\right) + \frac{m(m-1)}{1 \,.\, 2}\left(\frac{x}{a}\right)^2 + \&\text{c.}$$

$$= 1 + m\left(\frac{x}{a}\right) \text{ nearly}, = \frac{a + mx}{a} \text{ nearly}, = a + mx : a \text{ nearly}:$$

also, we shall similarly have

$$(a+x)^{\frac{1}{m}} : a^{\frac{1}{m}} = \left(\frac{a+x}{a}\right)^{\frac{1}{m}} = \left(1+\frac{x}{a}\right)^{\frac{1}{m}} = 1 + \frac{1}{m}\left(\frac{x}{a}\right)$$

$$+ \frac{1\,(1-m)}{1\,.\,2\,.\,m^2}\left(\frac{x}{a}\right)^2 + \&c. = 1 + \frac{1}{m}\left(\frac{x}{a}\right) \text{ nearly}, = \frac{a+\frac{x}{m}}{a} \text{ nearly},$$

$$= a + \frac{1}{m}x : a \text{ nearly}.$$

Hence, in cases of this kind, the ratios of the squares and square roots are respectively found by doubling and halving the difference.

Ex. We have therefore $(1002)^2 : (1000)^2 = 1004 : 1000$ nearly, and $\sqrt{1002} : \sqrt{1000} = 1001 : 1000$ nearly, the difference of these and the true values being in each case a very small fraction.

## II. Proportion.

237. Def. *Proportion* is the relation of equality expressed between two or more ratios, and is either *Arithmetical* or *Geometrical.*

Thus, if $a-b$ and $c-d$ be two ratios considered arithmetically, $a-b=c-d$ is termed an arithmetical proportion.

So also if $a : b$ and $c : d$, or $\frac{a}{b}$ and $\frac{c}{d}$ be two geometrical ratios, the equality

$$a : b = c : d,$$

(which is usually *written*, $a : b :: c : d$, and *read*, as $a$ is to $b$ so is $c$ to $d$)

$$\text{or } \frac{a}{b} = \frac{c}{d}$$

is styled a geometrical proportion.

As before, we shall here consider only the latter kind of proportion; and in this, $a$ and $d$ are called the *Extremes* and $b$ and $c$ the *Means*.

238. Cor. 1. If $a$, $b$, $c$ and $d$, taken in order be in geometrical proportion, we shall then have

$$a : b = c : d \text{ or } \frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} \times bd = \frac{c}{d} \times bd, \text{ or } ad = bc:$$

that is, the product of the extremes is equal to the product of the means.

This property enables us to convert a proportion into an equation, and the converse will manifestly be true; for,

$$\text{if } ad = bc,$$

$$\text{then } \frac{ad}{bd} = \frac{bc}{bd} \text{ or } \frac{a}{b} = \frac{c}{d},$$

$$\text{and } \therefore a : b :: c : d,$$

in which the factors of one member of the equation form the extremes, and those of the other the means.

239. Cor. 2. Hence if three quantities $a$, $b$, $c$ be in what is called *continued* proportion so that

$$a : b = b : c,$$

we shall have

$$ac = b^2,$$

or the product of the extremes is equal to the square of the mean, and conversely.

240. Cor. 3. From the equation $ad = bc$, we have

$$a = \frac{bc}{d}, \; b = \frac{ad}{c}, \; c = \frac{ad}{b} \text{ and } d = \frac{bc}{a};$$

and thus if any three terms in a proportion be given, the remaining one is found.

This corollary comprises the proof of what is called the *Single Rule of Three* in Arithmetic.

241. From what has been already said, it appears that the doctrine of proportion is merely the determination of the relations of fractions, whose numerators are the antecedents and denominators the consequents of the ratios which constitute them: therefore of the four quantities $a$, $b$, $c$, $d$ which form a proportion, there may be made various other arrangements and modifications in which proportionality will still be preserved.

Of these the most useful are the following:

(1). Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$, or $\frac{a}{c} = \frac{b}{d}$;

whence $a : c :: b : d$. (*Alternando*).

(2). Since $\frac{a}{b} = \frac{c}{d}$, $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$, or $\frac{b}{a} = \frac{d}{c}$;

whence $b : a :: d : c$. (*Invertendo*).

(3). Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$;

whence $a+b : b :: c+d : d$. (*Componendo*).

(4). Since $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$;

whence $a-b : b :: c-d : d$. (*Dividendo*).

(5). Since $\frac{a-b}{b} = \frac{c-d}{d}$ and $\frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a-b}{a} = \frac{c-d}{c}$;

whence $a-b : a :: c-d : c$. (*Convertendo*).

(6). Since $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a-b}{b} = \frac{c-d}{d}$, $\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}$; whence $a+b : a-b :: c+d : c-d$. (*Componendo and Dividendo*).

(7). Since $\frac{a}{b} = \frac{c}{d}$, $\therefore$ we have $\frac{ma}{mb} = \frac{nc}{nd}$, whence we obtain $ma : mb :: nc : nd$, where $m$ and $n$ may be either integral or fractional.

(8). Since $\frac{a}{b} = \frac{c}{d}$, we get $\frac{ma}{nb} = \frac{mc}{nd}$; whence we have $ma : nb :: mc : nd$, in which $m$ and $n$ may be either integral or fractional.

(9). Since $\frac{a}{b} = \frac{c}{d}$, we have $\left(\frac{a}{b}\right)^m = \left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m} = \frac{c^m}{d^m}$, whence $a^m : b^m :: c^m : d^m$, wherein $m$ may be integral or fractional.

242. Cor. If $a, b, c, d$ be proportionals, we have seen in (5) that

$$a-b : a :: c-d : c, \text{ or } \frac{a-b}{c-d} = \frac{a}{c}:$$

wherefore, if $a$ be the greatest term and consequently $d$ the least, by (238), we shall have

$$a - b > c - d,$$

also, $b + d = b + d$: whence by addition, we have

$$a + d > b + c:$$

or the greatest and the least together are greater than the other two together.

243. Similar considerations readily lead to the determination of the relations subsisting between different proportions, as will appear by the following instances.

(1). If $a : b :: c : d$ and $c : d :: e : f$, be two proportions, then since

$$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{c}{d} = \frac{e}{f},$$

$$\text{we have } \frac{a}{b} = \frac{e}{f};$$

$$\text{whence } a : b :: e : f;$$

and similar conclusions may be drawn if there be more proportions similarly connected.

(2). If $a : b :: c : d$ and $e : b :: f : d$, then we have

$$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{b} = \frac{f}{d},$$

$$\therefore \frac{a \pm e}{b} = \frac{c \pm f}{d};$$

which differently expressed becomes

$$a \pm e : b :: c \pm f : d.$$

(3). If $a, b, c; d, e, f$, be two sets of magnitudes, such that

$$a : b :: d : e$$

$$\text{and } b : c :: e : f;$$

then since

$$\frac{a}{b} = \frac{d}{e} \text{ and } \frac{b}{c} = \frac{e}{f},$$

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{d}{e} \times \frac{e}{f}, \text{ or } \frac{a}{c} = \frac{d}{f};$$

whence $a : c :: d : f$;

and similarly of more.

(4). If any number of magnitudes $a, b, c, d, e, f$, &c. be so circumstanced that $a : b :: c : d :: e : f ::$ &c., then since

$$\frac{a}{b} = \frac{a}{b}, \quad \frac{a}{b} = \frac{c}{d}, \quad \frac{a}{b} = \frac{e}{f}, \text{ \&c.}$$

we shall have $ab = ba$, $ad = bc$, $af = be$, &c.

$$\therefore\ ab + ad + af + \text{\&c.} = ba + bc + be + \text{\&c.}$$

$$\text{or } a(b + d + f + \text{\&c.}) = b(a + c + e + \text{\&c.}),$$

$$\text{and } \therefore\ \frac{a}{b} = \frac{a + c + e + \text{\&c.}}{b + d + f + \text{\&c.}}:$$

$$\text{whence } a : b :: a + c + e + \text{\&c.} : b + d + f + \text{\&c.}:$$

$$\text{similarly, } a : b :: a - c + e - \text{\&c.} : b - d + f - \text{\&c.};$$

and the converse of each manifestly holds good.

(5). If we have quantities in continued proportion so that

$$a : b :: b : c :: c : d :: \text{\&c.}$$

then by a process similar to the last, there will result

$a : b :: a + b + c + $ &c. $: b + c + d + $ &c.; and conversely.

(6). From the two proportions

$$a : b :: c : d$$

$$\text{and } e : f :: g : h,$$

$$\text{we get } \frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h};$$

$$\text{wherefore } \frac{ae}{bf} = \frac{cg}{dh}, \text{ by multiplication;}$$

$$\text{whence } ae : bf :: cg : dh:$$

that is, the products which arise from multiplying together the corresponding terms of the proportions are proportional: and it is obvious that the same holds good whatever number of proportions be supposed.

In the same manner, we should have by division,

$$\frac{a}{b} \div \frac{e}{f} = \frac{c}{d} \div \frac{g}{h} \quad \text{or} \quad \frac{af}{be} = \frac{ch}{dg};$$

$$\text{which gives } \frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}.$$

The former of these processes is called the *Compounding* of proportions, and contains the proof of *The Double Rule of Three* in Arithmetic.

244. Most of the results contained in articles (241) and (243) are of great practical utility, and are frequently enunciated at length so as to assume the form of rules.

245. If three magnitudes $a$, $b$, $c$ be in continued proportion so that

$$a : b :: b : c,$$

$$\text{then } \frac{a}{b} = \frac{b}{c};$$

$$\text{whence } \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2};$$

$$\therefore a : c :: a^2 : b^2,$$

or the first has to the third the duplicate ratio of what it has to the second.

Again, in four magnitudes whose relation is such that

$$a : b :: b : c :: c : d,$$

$$\text{we have } \frac{a}{b} = \frac{b}{c} = \frac{c}{d}:$$

$$\therefore \frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3};$$

$$\text{whence } a : d :: a^3 : b^3;$$

and so on, whatever be the number of magnitudes similarly circumstanced.

The two results here obtained constitute the *geometrical* definitions of duplicate and triplicate ratio.

246. When the four magnitudes $a$, $b$, $c$ and $d$ are proportional so that $a : b :: c : d$, we have seen that

$$\frac{ma}{nb} = \frac{mc}{nd}:$$

wherefore, if $ma$ be greater than, equal to, or less than $nb$, it follows that $mc$ will also be greater than, equal to, or less than $nd$.

Hence, of the terms of a proportion, there being taken any equimultiples *whatever* of the first and third, and any equimultiples *whatever* of the second and fourth, if the multiple of the first be greater than the multiple of the second, the multiple of the third will be greater than the multiple of the fourth; if equal, equal; and if less, less.

Also, conversely, if $a$, $b$, $c$, $d$ be four magnitudes so circumstanced that $ma$ is always greater than, equal to, or less than $nb$, according as $mc$ is greater than, equal to, or less than $nd$, then will $a$, $b$, $c$ and $d$, taken in order, be the terms of a proportion:

for if not, let $a$, $b$, $c$ and $e$ be such terms: then, since $a : b :: c : e$, we have

$$\frac{ma}{nb} = \frac{mc}{ne},$$

whatever be the values of $m$ and $n$: therefore, if $ma$ be greater than, equal to, or less than $nb$, it will follow that $mc$ is greater than, equal to, or less than $ne$: but the same having been asserted of $nd$, we shall have

$$ne = nd, \text{ and } \therefore\ e = d:$$

whence $a : b :: c : d$, or the magnitudes $a$, $b$, $c$ and $d$ are the terms of a proportion.

Similarly, if what has been enunciated above do not obtain in the magnitudes $a$, $b$, $c$ and $d$, it may be made to appear that they cannot in order form the terms of a proportion.

247. The *Algebraical* characteristic of proportionality proved in the last article, is manifestly applicable to all kinds of magnitudes whatever; and it is found to agree with the *Geometrical* definition of proportion laid down by *Euclid* in the fifth, &c. definitions of the fifth book of the Elements.

Number being a *discrete*, and extension a *continuous* magnitude, it is obvious that the parts of number will be more distinct, and on that account more easily assignable than the parts of extension; and this may be surmised to be the reason why, as a test of *geometrical* proportionality, recourse has been had to the use of *multiples*, instead of *aliquot parts*, which have been adopted as the basis of the *algebraical* and *arithmetical* views of the subject.

248. Taking the three quantities $a$, $b$, $c$, we have seen that an *Arithmetical* proportion subsists among them when

$$a - b = b - c, \text{ and } \therefore\ 1 \text{ or } \frac{a}{a} = \frac{a-b}{b-c};$$

$$\text{so that } a : a :: a - b : b - c:$$

also, if they be in *Geometrical* proportion, their relation is such that

$$\frac{a}{b} = \frac{b}{c}, \text{ and } \therefore\ \frac{a-b}{a} = \frac{b-c}{b}, \text{ or } \frac{a}{b} = \frac{a-b}{b-c};$$

$$\text{so that } a : b :: a - b : b - c:$$

and it is observable, that in these two proportions the consequents of the first ratios are $a$ and $b$ respectively: whence it manifestly follows, that if $c$ become the consequent of the said ratio, we shall have a proportion different from them both, namely,

$$a : c :: a - b : b - c:$$

and this, from certain properties which it possesses, is called a *Harmonical* proportion.

249. Cor. By means of these three proportions the values of the *Arithmetic*, *Geometric* and *Harmonic* means between the two quantities $a$ and $c$ may immediately be found. Thus,

$$\text{the arithmetic mean} = \frac{a+c}{2};$$

$$\text{the geometric mean} = \sqrt{ac};$$

$$\text{the harmonic mean} = \frac{2ac}{a+c}:$$

$$\text{and since } \frac{a+c}{2} : \sqrt{ac} :: \sqrt{ac} : \frac{2ac}{a+c},$$

it appears that the three means taken in this order form a geometrical proportion:

also since by (22), $(a-c)^2$ is always positive and therefore greater than 0, we have

$$a^2 + 2ac + c^2 > 4ac;$$

$$\text{whence } \frac{a+c}{2} > \sqrt{ac};$$

or the first mean is greater than the second, and the second therefore greater than the third.

## III. Variation.

250. Def. A quantity is said to *vary* as one or more others, when it is so dependent upon them, that every change which they undergo, produces a corresponding and *proportional* change in its magnitude; and it is consequently connected with them by some multiplier, either integral or fractional, which remains the same during the whole of any operation in which they are concerned.

The different kinds of *Variation* are distinguished as follows, the symbol $\propto$ expressing this connection.

(1). If $A = pB$, $A$ varies *directly* as $B$, or $A \propto B$:

(2). If $A = \dfrac{p}{B}$, $A$ varies *inversely* as $B$, or $A \propto \dfrac{1}{B}$:

(3). If $A = pBC$, $A$ varies as $B$ and $C$ *jointly*, or $A \propto BC$:

(4). If $A = p\dfrac{B}{C}$, $A$ varies as $B$ *directly* and $C$ *inversely*,

$$\text{or} \quad A \propto \frac{B}{C}: \quad \&c.$$

and the same may be extended to more magnitudes.

It is obvious that the variation here intended is merely an abbreviation of the doctrine of proportion before explained: for if we have the proportion

$$A : B :: a : b,$$

then will $A = \dfrac{a}{b} B$, in which $\dfrac{a}{b}$ may be represented by the invariable quantity $p$ above used: so of others.

251. The doctrine of equations, as laid down in the preceding pages, will lead immediately to all the consequences which the view of variation above adopted presents.

(1). *If* A $\propto$ B *and* B $\propto$ C, *then will* A $\propto$ C.

$$\text{For, if } A = pB \text{ and } B = qC;$$

we shall have $A = pB = pqC$; that is, $A \propto C$.

$$\text{Hence also, if } A \propto \frac{1}{B} \text{ and } B \propto C,$$

$$\text{then } A = \frac{p}{B} \text{ and } B = qC;$$

$$\text{whence we have } A = \frac{p}{B} = \frac{p}{q}\,\frac{1}{C}, \text{ or } A \propto \frac{1}{C}.$$

(2). *If* A $\propto \frac{1}{\mathrm{B}}$ *and* B $\propto \frac{1}{\mathrm{C}}$, *then will* A $\propto$ C.

$$\text{For, if } A = \frac{p}{B} \text{ and } B = \frac{q}{C};$$

$$\text{we have } A = \frac{p}{B} = \frac{p}{q}\,C; \text{ that is, } A \propto C:$$

and in the same manner whatever be the number of magnitudes, when each varies inversely as the following, the first varies directly or inversely as the last, according as the number of intermediate magnitudes is odd or even.

(3). *If* A $\propto$ C *and* B $\propto$ C, *then will* A $\pm$ B $\propto$ C *and* $\sqrt{\mathrm{AB}} \propto$ C.

$$\text{For, if } A = pC \text{ and } B = qC;$$

$$\text{then } A \pm B = (p \pm q)\,C, \text{ and } \therefore\ A \pm B \propto C:$$

also, $AB = pqC^2$, and $\therefore\ \sqrt{AB} = \sqrt{pq}\,C$, or $\sqrt{AB} \propto C$:

and similar conclusions may be drawn whatever be the number of quantities concerned.

(4). *If* $A \propto B$, *then will* $AP \propto BP$, *and* $\frac{A}{P} \propto \frac{B}{P}$, *where* P *may be either variable or invariable.*

For, if $A = pB$, we have $AP = pBP$ and $\frac{A}{P} = p\frac{B}{P}$; whence it follows that $AP \propto BP$ and $\frac{A}{P} \propto \frac{B}{P}$.

Hence also, $A^m = p^m B^m$, and therefore $A^m \propto B^m$, where $m$ may be either integral or fractional.

(5). *If* $A \propto BC$, *then will* $B \propto \frac{A}{C}$ *and* $C \propto \frac{A}{B}$.

For, if we take $A = pBC$, then will

$$B = \frac{1}{p}\frac{A}{C} \propto \frac{A}{C}, \text{ and } C = \frac{1}{p}\frac{A}{B} \propto \frac{A}{B}.$$

Hence also, if $A$ be invariable and equal to $q$, then

$$B = \frac{q}{p}\frac{1}{C} \text{ or } \propto \frac{1}{C}, \text{ and } C = \frac{q}{p}\frac{1}{B} \text{ or } \propto \frac{1}{B}.$$

(6). *If* $A \propto B$ *and* $C \propto D$, *then will* $AC \propto BD$ *and* $\frac{A}{C} \propto \frac{B}{D}$.

For, if $A = pB$ and $C = qD$; $\therefore AC = pqBD$, or $AC \propto BD$;

$$\text{also, } \frac{A}{C} = \frac{p}{q}\frac{B}{D}, \text{ or } \frac{A}{C} \propto \frac{B}{D}.$$

Similar results will be obtained whatever be the number of quantities employed.

(7). *If* $A \propto B$ *when* C *is invariable, and* $A \propto C$ *when* B *is invariable, then will* $A \propto BC$, *when both* B *and* C *are variable.*

For, we may have $A = pCB$ and $A = qBC$; whence

$$A^2 = pq(BC)^2, \text{ or } A \propto BC:$$

and similarly whatever be the number of latter magnitudes as $B$, $C$, $D$, &c.

(8). From the proportion $A : B :: C : D$, we have

$$A = \frac{BC}{D}:$$

and $\therefore$ $A \propto BC$, when $D$ is given;

$A \propto \frac{B}{D}$, when $C$ is given;

and $A \propto \frac{1}{D}$, when $B$ and $C$ are given.

252. In discussing the subjects of the present Chapter, it has been supposed that all the quantities concerned are some multiples, part or parts of one another, and that their relations to each other may therefore be expressed by means of whole numbers: in fact, they have been supposed to have at *least* unity for their common measure, or, in other words, to be *commensurable*.

Thus, if the ratio $a : b$ were $2 : 1$, which may be also expressed *symbolically* by $2\sqrt{2} : \sqrt{2}$ or $\sqrt{8} : \sqrt{2}$, such a ratio, being *commensurable*, is the subject of the operations and observations contained in the preceding articles.

Should, however, the ratio $a : b$ be $\sqrt{2} : 1$, then, since by (128) we have seen that $\sqrt{2} = 1.4142135$ &c. *in infinitum*, it follows that

$$\begin{aligned}\sqrt{2} : 1 &= 14 : 10, \text{ nearly};\\ &= 141 : 100, \text{ more nearly};\\ &= 1414 : 1000, \text{ still more nearly};\\ &= \text{\&c.} \ldots\ldots\ldots\ldots\end{aligned}$$

and as this may be continued in *infinitum*, it is manifest that the proposed ratio can never be exactly expressed by

any number of the parts of an unit, be they ever so small. This ratio is therefore *incommensurable*, though limits may be found and expressed, between which it shall always consist, and by which it may be exhibited to any degree of exactness required.

Hence then, it follows, that the algebraical definition and criterion of proportionality for commensurable magnitudes, will not be sufficient for our idea of proportionality subsisting among incommensurable quantities. A characteristic of proportionality has indeed been established in (246), which relates equally to commensurable and incommensurable magnitudes: but since the values of incommensurable quantities may by (127) be exhibited to any required degree of accuracy, it may thence be readily shewn that the doctrines of ratio and proportion hold good, whatever be the nature of the quantities among which they are instituted.

For, let $a$ and $b$ be incommensurable magnitudes, which admit of no common measure whatever, and suppose $b = nx$ and $a$ to lie between $mx$ and $(m+1)x$: then is the ratio

$$\frac{a}{b} > \frac{m}{n} \text{ but } < \frac{m+1}{n}; \text{ or } \frac{a}{b} - \frac{m}{n} < \frac{1}{n},$$

which by the diminution of $x$ may obviously be made less than any quantity that can be assigned: and therefore whatever is proved of the ratio $m : n$ in this case, holds also of the ratio $a : b$.

Again, if $\frac{a}{b}$ and $\frac{c}{d}$ represent two incommensurable ratios, which can both be proved to lie between $\frac{m}{n}$ and $\frac{m+1}{n}$, whatever be the magnitudes of $m$ and $n$, we shall have

$$\frac{a}{b} - \frac{c}{d} < \frac{1}{n},$$

and therefore, by reasoning as before, $\dfrac{a}{b} = \dfrac{c}{d}$;

or the incommensurable ratio $a : b$ is equal to the incommensurable ratio $c : d$; and thus proportionality is established among the incommensurable magnitudes $a$, $b$, $c$ and $d$.

Without, however, endeavouring to obtain their approximate values, we may easily shew how incommensurable ratios may be compared with each other: thus, if the ratios be

$$\sqrt[3]{3} : \sqrt{2} \text{ and } \sqrt{3} : \sqrt[3]{5};$$

we have the former $= \dfrac{3^{\frac{1}{3}}}{2^{\frac{1}{2}}} = \dfrac{3^{\frac{2}{6}}}{2^{\frac{3}{6}}} = \left(\dfrac{9}{8}\right)^{\frac{1}{6}} = \left(\dfrac{27}{24}\right)^{\frac{1}{6}}$,

and the latter $= \dfrac{3^{\frac{1}{2}}}{5^{\frac{1}{3}}} = \dfrac{3^{\frac{3}{6}}}{5^{\frac{2}{6}}} = \left(\dfrac{27}{25}\right)^{\frac{1}{6}}$;

from which we immediately conclude that the former ratio is greater than the latter.

**253.** Subjoined are a few problems together with their solutions, wherein the principles explained in this chapter are called into use.

(1). To find two magnitudes having the ratio of $m : n$, so that if the given quantity $a$ be added to each, the sums shall have the ratio of $p : q$.

Let $mx$ and $nx$, having the proposed ratio, denote the required quantities:

then will $\dfrac{mx + a}{nx + a} = \dfrac{p}{q}$ by the question:

whence $x = \dfrac{(p - q)\,a}{mq - np}$:

and the magnitudes sought will be respectively

$$\frac{(p-q)\,ma}{mq-np} \text{ and } \frac{(p-q)\,na}{mq-np}.$$

(2). Required two numbers in the ratio of 4 : 5, from which, if two other required numbers in the ratio of 6 : 7 be respectively subtracted, the remainders shall be in the ratio of 2 : 3, and their sum equal to 20.

Let $4x$ and $5x$ be the first two numbers:

$6y$ and $7y$, the other two:

then $\dfrac{4x-6y}{5x-7y}=\dfrac{2}{3}$, and $9x-13y=20$, by the question:

whence are easily found $y=4$ and $x=8$, so that the first two numbers are 32 and 40, and the other two 24 and 28.

(3). Given two magnitudes $a$ and $b$, to find two other equal magnitudes, so that the ratio of the sums of each two may be equal to the ratio of their products respectively.

Let $x$ and $x$ represent the required magnitudes; then by the question we have

$$\frac{a+b}{2x}=\frac{ab}{x^2};$$

whence $x=\dfrac{2ab}{a+b}=$ each of the magnitudes sought, as may easily be verified.

Similarly, if three or more magnitudes be proposed.

(4). To divide each of the quantities $a$, $b$ and $c$ into two parts, so that the ratio of the second part of $a$ to the first part of $b$ may be $m$ : 1: that of the second part of $b$ to the first part of $c$ may be $n$ : 1: and that of the second part of $c$ to the first part of $a$ may be $p$ : 1.

Let $x$ be the first part of $a$, and therefore $a-x$ the second part:

$$\therefore \frac{a-x}{\text{first part of } b}=\frac{m}{1}, \text{ and the first part of } b=\frac{1}{m}(a-x):$$

$$\text{whence the second part of } b=b-\frac{1}{m}(a-x):$$

$$\therefore \frac{b-\frac{1}{m}(a-x)}{\text{first part of } c}=\frac{n}{1}, \text{ and the first part of } c=\frac{1}{n}\left\{b-\frac{1}{m}(a-x)\right\};$$

$$\text{wherefore the second part of } c=c-\frac{1}{n}\left\{b-\frac{1}{m}(a-x)\right\}:$$

$$\therefore \frac{c-\frac{1}{n}\left\{b-\frac{1}{m}(a-x)\right\}}{x}=\frac{p}{1}, \text{ by the question:}$$

$$\text{whence } x=\frac{a-mb+mnc}{1+mnp},$$

and the respective parts of $a$, $b$ and $c$ may therefore easily be found.

A similar method may obviously be pursued whatever be the number of magnitudes and ratios proposed.

(5). Divide the quantity $a$ into two parts, so that their difference may be to their sum as their product to the difference of their squares.

Let $x$ and $y$ denote the parts required; then we have by the question

$$x+y=a,$$

$$\text{and } x-y : x+y :: xy : x^2-y^2:$$

whence, by multiplying together the extremes and means respectively, we obtain the equation

$$(x-y)(x^2-y^2)=(x+y)\,x\,y;$$

$$\text{wherefore } (x-y)^2=xy \text{ and } (2x-a)^2=x(a-x):$$

from which we readily find

$$x = \left(\frac{5 \pm \sqrt{5}}{10}\right) a, \text{ and } \therefore\ y = \left(\frac{5 \mp \sqrt{5}}{10}\right) a.$$

(6). Decompose the quantity $a^2$ into two factors, so that the sum of their cubes may be to the difference of their cubes as $m : 1$.

Assuming $x$ and $y$ to represent the factors required, we have

$$xy = a^2,$$

$$\text{and } x^3 + y^3 : x^3 - y^3 :: m : 1, \text{ by the question;}$$

$$\text{whence } 2x^3 : 2y^3 :: m+1 : m-1, \text{ by (6) of Art. (241),}$$

$$\text{and } x^3 : y^3 :: m+1 : m-1, \text{ by (7) } \ldots\ldots\ldots\ldots\ldots$$

therefore, multiplying extremes and means, we find

$$(m-1)\,x^3 = (m+1)\,y^3 = (m+1)\frac{a^6}{x^3}:$$

$$\therefore\ x = \pm a \sqrt[6]{\frac{m+1}{m-1}} \text{ and } y = \pm a \sqrt[6]{\frac{m-1}{m+1}}.$$

(7). A waterman rows a given distance $a$ and back again in $b$ hours, and finds that he can row $c$ miles with the tide for $d$ miles against it: required the rate of the tide and the times of rowing down and up the stream.

Let $x =$ the number of hours he rows with the tide,

then will $b - x =$ the time he rows against it:

$$\text{also, } x : 1 :: a : \frac{a}{x} = \text{rate per hour with the tide:}$$

$$\text{and } b - x : 1 :: a : \frac{a}{b-x} = \text{rate per hour against it:}$$

$$\text{whence } \frac{a}{x} : \frac{a}{b-x} :: c : d, \text{ by the question:}$$

$$\text{and } \therefore\ x = \frac{bd}{c+d} = \text{the time with the tide:}$$

$$\text{and } b - x = \frac{bc}{c+d} = \text{the time against the tide:}$$

now $a \div \dfrac{bd}{c+d} = \dfrac{a(c+d)}{bd}$ = the rate down the stream = the rate of rowing + the rate of the tide:

also, $a \div \dfrac{bc}{c+d} = \dfrac{a(c+d)}{bc}$ = the rate up the stream = the rate of rowing − the rate of the tide:

whence $\dfrac{a(c+d)}{bd} - \dfrac{a(c+d)}{bc}$ = twice the rate of the tide;

therefore the rate of the tide $= \dfrac{a(c^2 - d^2)}{2bcd}$.

(8). If from a cask of wine containing $a$ gallons, $b$ gallons be drawn off and the vessel filled up with water, and this be repeated $n$ times successively, find the quantity of wine then remaining.

Let $a_1$, $a_2$, $a_3$, &c. $a_n$ denote the quantities of wine remaining after the operation has been repeated once, twice, thrice, &c., $n$ times respectively: then it is obvious that

$$a : a_1 :: a : a - b:$$

but since the strength of the mixture, and therefore the wine in it, manifestly decreases at every operation in the ratio of $a : a - b$, we have

$$a_1 : a_2 :: a : a - b,$$
$$a_2 : a_3 :: a : a - b,$$
&c.......................
$$a_{n-1} : a_n :: a : a - b:$$

whence, by the composition of these equal ratios, is obtained

$$a : a_n :: a^n : (a-b)^n,$$

$$\text{and therefore } a_n = \left(\frac{a-b}{a}\right)^n a = \frac{(a-b)^n}{a^{n-1}}.$$

(9). If $a$ men can reap a rectangular field of wheat, whose length and breadth are $b$ and $c$ chains respectively, in a certain time, what number of men will it require to reap one $d$ chains long and $e$ chains broad in the same time?

Here it is manifest that the numbers of men will be in the same ratio as the extents of the fields: and the fields are obviously in the ratio compounded of those of their respective lengths and breadths, that is, in the ratio of $bc : de$:

whence if $x$ denote the number of men required, we shall have

$$bc : de :: a : x,$$

wherefore $x = \frac{ade}{bc}$ = the number of men required.

This is an illustration of the *Rule of Three Direct*, the effect produced in a certain time being *directly* proportional to the number of agents employed.

(10). If a person make a journey of $a$ miles in $b$ days, when a day is $c$ hours in length, in how many days can he perform the same when the days are $d$ hours long?

Let $x$ represent the number of days required; then it is manifest that the lengths of his journies will be in the ratio compounded of those of the numbers and lengths of the days: wherefore in this case we have

$$a : a :: bc : dx \text{ or } bc = dx,$$

whence we obtain $x = \frac{bc}{d}$, which is therefore the fourth term of the proportion

$$d : c :: b : x.$$

This illustrates the *Rule of Three Inverse*, the numbers of days being *inversely* proportional to their lengths.

(10). If $a$ horses eat up $b$ bushels of oats in $c$ days: how many horses will eat up $d$ bushels in $e$ days?

Let $x$ be taken to denote the required number; then it is clear from (7) of Art. (251) that the quantity of oats eaten will vary as the numbers of horses and days conjointly: whence

$$b : d :: ac : ex,$$

and therefore $x = \dfrac{acd}{be}$ = the number of horses required.

Here $\dfrac{b}{d} : \dfrac{c}{e} :: a : x$, which being put in the form following,

$$\begin{matrix} b : c :: a \\ d : e :: x \end{matrix}$$

exemplifies the application of what is usually called the *Double Rule of Three.*

(11). $A$, $B$ and $C$ hold a pasture in common, for which they pay $P$£. per annum: $A$ puts into it $a$ oxen for $m$ months; $B$, $b$ oxen for $n$ months, and $C$, $c$ oxen for $p$ months: required the share of the rent contributed by each.

The contributions of $A$, $B$ and $C$ will obviously be to each other as the quantities $ma$, $nb$ and $pc$ respectively; and it therefore remains to divide $P$£. into three parts having to each other the same ratios; whence if $x$ represent $A$'s share of the rent, we have

$$ma : nb :: x : \frac{nbx}{ma} = B\text{'s share,}$$

$$\text{and } ma : pc :: x : \frac{pcx}{ma} = C\text{'s share;}$$

$$\therefore\ x + \frac{nbx}{ma} + \frac{pcx}{ma} = P, \text{ by the question:}$$

$$\text{whence } x = \frac{ma}{ma+nb+pc} P£. = A\text{'s share;}$$

$$\frac{nbx}{ma} = \frac{nb}{ma+nb+pc} P£. = B\text{'s share;}$$

$$\text{and } \frac{pcx}{ma} = \frac{pc}{ma+nb+pc} P£. = C\text{'s share.}$$

This is the investigation of the rule in an instance of what is called *Double Fellowship* or *Fellowship with Time:*

and if $m=n=p$, or the time be the same for each, we find

$$A\text{'s share} = \frac{a}{a+b+c} P£.;$$

$$B\text{'s share} = \frac{b}{a+b+c} P£.;$$

$$\text{and } C\text{'s share} = \frac{c}{a+b+c} P£.:$$

which is an example, wherein is deduced the rule for *Single Fellowship* or *Fellowship without Time.*

(12). A mixture is made of $a$ lbs. of tea at $m$ shillings per lb., $b$ lbs. at $n$ shillings, and $c$ lbs. at $p$ shillings: what will be its price per lb.?

Let $x$ denote the price required; then since the price of the whole varies as the number of lbs. and the price per lb. conjointly, we shall have

$$ma+nb+pc = \text{the sum of the prices of the ingredients:}$$

$$\text{and } (a+b+c)\,x = \text{the price of the mixture:}$$

whence by the question, we must have

$$(a+b+c)\,x = ma + nb + pc,$$

$$\text{and } \therefore\ x = \frac{ma+nb+pc}{a+b+c}.$$

In this manner is investigated the arithmetical rule of *Alligation Medial*, or the method of finding the rate or quality of the composition from the rates or qualities of the ingredients.

(13). If $a$ oxen in $m$ weeks eat $b$ acres of grass, and $c$ oxen eat $d$ acres in $n$ weeks, how many oxen will eat $e$ acres in $p$ weeks, the grass being supposed to grow uniformly?

Put $x$ for the number of oxen sought:

let $\alpha =$ the grass upon an acre at first,

and $\beta =$ the increase of grass upon an acre in a week:

$\therefore\ \alpha + m\beta =$ the grass on an acre in $m$ weeks:

$\alpha + n\beta =$ ......................... $n$ ......;

$\alpha + p\beta =$ ......................... $p$ ......:

$\therefore\ b(\alpha + m\beta) =$ the grass on $b$ acres in $m$ weeks;

$d(\alpha + n\beta) =$ ............. $d$ ......... $n$ ......;

$e(\alpha + p\beta) =$ ............. $e$ ......... $p$ ......:

now it is manifest that the quantity of grass consumed will, *cæteris paribus*, vary as the number of oxen: therefore we shall have

$$b(\alpha + m\beta) : d(\alpha + n\beta) :: a : c,$$

and

$$b(\alpha + m\beta) : e(\alpha + p\beta) :: a : x:$$

from the former, we find

$$bc(\alpha + m\beta) = ad(\alpha + n\beta), \text{ and } \therefore\ \beta = \frac{(ad-bc)\alpha}{mbc-nad}:$$

and from the latter we obtain

$$x = \frac{ae(\alpha + p\beta)}{b(\alpha + m\beta)} = \frac{ae}{b}\left\{\frac{\alpha + \dfrac{p(ad-bc)}{mbc-nad}\alpha}{\alpha + \dfrac{m(ad-bc)}{mbc-nad}\alpha}\right\}$$

$$= \frac{ae}{b}\left\{\frac{mbc-nad+pad-pbc}{mbc-nad+mad-mbc}\right\}$$

$$= \frac{ae}{b}\left\{\frac{(m-p)bc-(n-p)ad}{(m-n)ad}\right\}$$

$$= \frac{e}{bd}\left\{\frac{(m-p)bc-(n-p)ad}{m-n}\right\}$$

$$= \left(\frac{m-p}{m-n}\right)\frac{ce}{d} - \left(\frac{n-p}{m-n}\right)\frac{ae}{b}:$$

from which may be deduced the following practical formula:

$$(m-n)\frac{x}{e} = (m-p)\frac{c}{d} - (n-p)\frac{a}{b}.$$

# CHAP. IX.

## *On Arithmetical, Geometrical and Harmonical Progression.*

### I. Arithmetical Progression.

254. Def. An *Arithmetical Progression* is a series of three or more quantities in continued arithmetical proportion, its characteristic property therefore being, that every term exceeds or falls short of that which immediately precedes it by the same *Common Difference*.

Thus, $a$, $2a$, $3a$, &c.; $a$, $a+b$, $a+2b$, &c.; $a$, $a-2x$, $a-4x$, &c., are all arithmetical progressions, the two former increasing by the common differences $a$ and $b$ respectively, and the last decreasing by the common difference $2x$: in the first two instances the common difference is positive, and in the last it is negative.

255. Cor. Hence the terms of an arithmetical progression taken at equal intervals are also in arithmetical progression.

256. *Given the first term and the common difference of an arithmetical progression, to find the* n$^{\text{th}}$ *term and the sum of* n *terms.*

Let $a$ be the first term, $d$ the common difference, $l$ the $n^{\text{th}}$ term and $S$ the sum of $n$ terms:

then will $a$, $a+d$, $a+2d$, $a+3d$, &c. be the series:

and since the first term does not involve $d$, but the second does, it is evident that $(n-1)\,d$ will be the multiple of $d$ found in the $n^{\text{th}}$ term, and thence we shall have

$$l = a + (n-1)\,d.$$

Again, $S = a + (a+d) + (a+2d) + \&c.$ to $n$ terms

$$= a + (a+d) + (a+2d) + \&c. + (l-2d) + (l-d) + l;$$

and reversing the order of the terms, we have also

$$S = l + (l-d) + (l-2d) + \&c. + (a+2d) + (a+d) + a;$$

$\therefore$ by adding together the corresponding terms, we obtain

$$2S = (a+l) + (a+l) + (a+l) + \&c. \text{ to } n \text{ terms}$$

$$= (a+l)\,n:$$

$$\therefore\ S = (a+l)\frac{n}{2} = \{a + a + (n-1)d\}\frac{n}{2} = \{2a + (n-1)d\}\frac{n}{2}.$$

If $d$ be negative, or the series a decreasing one, we shall have

$$S = \{2a - (n-1)\,d\}\,\frac{n}{2};$$

and it is obvious that the sum of the series may vanish either by making $n = 0$; or by assuming $2a - (n-1)\,d = 0$, which gives

$$n = \frac{2a+d}{d} = 1 + \frac{2a}{d}.$$

Ex. 1. To find the $n^{\text{th}}$ term and the sum of $n$ terms of the series of odd numbers, 1, 3, 5, 7, &c.

Generally, $l = a + (n-1)\,d$, and $S = \{2a + (n-1)\,d\}\frac{n}{2}$:

and here $a = 1$, $d = 2$; whence by substitution, we get

$$l = 1 + (n-1)\,2 = 2n - 1,$$

$$\text{and } S = \{2 + (n-1)\,2\}\,\frac{n}{2} = n^2:$$

that is, the $n^{\text{th}}$ of the odd natural numbers beginning with unity is expressed by $2n-1$, and the sum of the first $n$ of them by $n^2$.

Ex. 2. To find the $n^{\text{th}}$ term and the sum of $n$ terms of the series of even numbers, 2, 4, 6, 8, &c.

Here $a = 2$ and $d = 2$; wherefore, as before, we obtain

$$l = 2 + (n-1)2 = 2n,$$

$$\text{and } S = \{4 + (n-1)2\}\frac{n}{2} = n(n+1):$$

or, the $n^{\text{th}}$ even natural number is $2n$, and the sum of the first $n$ such numbers is $n(n+1)$.

Ex. 3. Required the $n^{\text{th}}$ term and the sum of $n$ terms of the series $(b+x)^2$, $b^2+x^2$, $(b-x)^2$, &c.

Here $a = (b+x)^2$ and $d = -2bx$; whence we have

$$\begin{aligned} l &= (b+x)^2 - (n-1)2bx \\ &= b^2 + x^2 - (2n-4)bx: \end{aligned}$$

$$\begin{aligned} \text{and } S &= \{2(b+x)^2 - (n-1)2bx\}\frac{n}{2} \\ &= \{(b+x)^2 - (n-1)bx\}n \\ &= \{b^2 - (n-3)bx + x^2\}n \\ &= n(b^2+x^2) - n(n-3)bx. \end{aligned}$$

257. Cor. In the two fundamental equations:

$$l = a + (n-1)d,$$

$$\text{and } S = \{2a + (n-1)d\}\frac{n}{2},$$

if any three of the quantities involved be given, the remaining one may be found by the solution of the equations with respect to it; and from the two equations combined, it will not be difficult to arrive at the following results:

(1). $a = l - (n-1)d = \frac{2S}{n} - l = \frac{S}{n} - \frac{(n-1)d}{2}$

$$= \frac{d}{2} \pm \sqrt{l^2 + ld + \frac{d^2}{4} - 2Sd}.$$

(2). $l = a + (n-1)d = \frac{2S}{n} - a = \frac{S}{n} + \frac{(n-1)d}{2}$

$$= -\frac{d}{2} \pm \sqrt{a^2 - ad + \frac{d^2}{4} + 2Sd}.$$

(3). $d = \frac{l-a}{n-1} = \frac{2(S-an)}{n(n-1)} = \frac{2(ln-S)}{n(n-1)} = \frac{l^2 - a^2}{2S - a - l}.$

(4). $n = 1 + \frac{l-a}{d} = \frac{2S}{a+l} = \frac{1}{2} - \frac{a}{d} \pm \sqrt{\frac{a^2}{d^2} - \frac{a}{d} + \frac{1}{4} + \frac{2S}{d}}$

$$= \frac{1}{2} + \frac{l}{d} \pm \sqrt{\frac{l^2}{d^2} + \frac{l}{d} + \frac{1}{4} - \frac{2S}{d}}.$$

(5). $S = \frac{n(a+l)}{2} = na + \frac{n(n-1)d}{2} = nl - \frac{n(n-1)d}{2}$

$$= \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$$

Ex. Let $a = 7$, $d = 2$ and $S = 40$, then from (4) of this article, we have

$$n = \frac{1}{2} - \frac{7}{2} \pm \sqrt{9 + 40} = 4 \text{ and } -10,$$

of which the latter is excluded by the nature of the case (166): and the series corresponding to the first is

7, 9, 11, 13.

Here it may be remarked that, though in the ordinary acceptation of the terms the latter value of $n$ is excluded, there is still a series corresponding to it: for if $n$ be negative, we have

$$S = \left\{-2a + (n+1)d\right\}\frac{n}{2}$$

$$= n(n-6) \text{ in this case;}$$

therefore the term which begins the series will obviously be its sum when $n=1$; that is, the term beginning the series is $-5$, and the series itself will therefore be

$$-5,\ -3,\ -1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,$$

which clearly answers the specified conditions: and generally, corresponding to a negative value of $n$, the first term becomes $d-a$, and the series will then be

$$d-a,\ 2d-a,\ 3d-a,\ \&c.$$

258. *If the two extremes and the number of terms be given, the series may be found.*

For, since $l = a + (n-1)d$, we have $d = \dfrac{l-a}{n-1}$, which is the common difference; and therefore the series is

$$a,\ \frac{(n-2)a+l}{n-1},\ \frac{(n-3)a+2l}{n-1},\ \&c.,\ \frac{2a+(n-3)l}{n-1},\ \frac{a+(n-2)l}{n-1},\ l.$$

259. Cor. Hence $m$ arithmetic means or intermediate terms may be inserted between $a$ and $l$.

For, since the number of terms exceeds the number of means by 2, we have $n = m+2$, and $\therefore n-1 = m+1$; whence the $m$ means are

$$\frac{ma+l}{m+1},\ \frac{(m-1)a+2l}{m+1},\ \&c.,\ \frac{2a+(m-1)l}{m+1},\ \frac{a+ml}{m+1}.$$

Ex. Find two arithmetic means between $-3$ and $3$.

Here $a = -3$, $l = 3$ and $m = 2$, which give the means equal

$$\text{to } \frac{-2.3+3}{2+1} = -1 \text{ and } \frac{-1.3+2.3}{2+1} = 1;$$

and $-3$, $-1$, $1$ and $3$ are evidently in arithmetical progression.

260. *If the magnitudes of any two terms whose places are known, be given, the series may be determined.*

For, let $P$ and $Q$ represent the $p^{\text{th}}$ and $q^{\text{th}}$ terms respectively, and the rest be as before: then we have by (256),

$$P = a + (p-1)d, \text{ and } Q = a + (q-1)d;$$

$$\text{whence } P - Q = (p-q)d, \text{ and } \therefore d = \frac{P-Q}{p-q}:$$

$$\therefore a = P - (p-1)d = P - \left(\frac{p-1}{p-q}\right)(P-Q)$$

$$= \frac{Q(p-1) - P(q-1)}{p-q}:$$

therefore the first term and common difference being found, the series becomes known.

Also, the $n^{\text{th}}$ term $= a + (n-1)d$

$$= \frac{Q(p-n) - P(q-n)}{p-q};$$

and the sum of $n$ terms $= \{2a + (n-1)d\} \dfrac{n}{2}$

$$= \left\{\frac{Q(2p-n-1) - P(2q-n-1)}{p-q}\right\} \frac{n}{2}.$$

261. *To find the sums of the powers of the terms of an arithmetical progression.*

Let the terms of the series be $a$, $b$, $c$, &c. $k$, $l$, and the common difference $d$ as before: also let $S_1$, $S_2$, $S_3$, &c. $S_{m-1}$, denote the sums of the 1st, 2nd, 3rd, &c., $(m-1)$th powers, of the terms: then we have by the binomial theorem

$$b^m - a^m = (a+d)^m - a^m = m a^{m-1} d + \frac{m(m-1)}{1 \cdot 2} a^{m-2} d^2 + \text{\&c.}$$

$$c^m - b^m = (b+d)^m - b^m = m b^{m-1} d + \frac{m(m-1)}{1 \cdot 2} b^{m-2} d^2 + \text{\&c.}$$

&c. ........................................................

$$l^m - k^m = (k+d)^m - k^m = m k^{m-1} d + \frac{m(m-1)}{1 \cdot 2} k^{m-2} d^2 + \text{\&c.}$$

$$\text{and } (l+d)^m - l^m = m l^{m-1} d + \frac{m(m-1)}{1 \cdot 2} l^{m-2} d^2 + \text{\&c.};$$

$\therefore$ by the addition of these lines in vertical rows, we obtain

$$(l+d)^m - a^m = m S_{m-1} d + \frac{m(m-1)}{1 \cdot 2} S_{m-2} d^2 + \text{\&c.}:$$

whence is immediately deduced

$$m S_{m-1} d =$$

$$(l+d)^m - a^m - \frac{m(m-1)}{1 \cdot 2} S_{m-2} d^2 - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} S_{m-3} d^3 - \text{\&c.}$$

$$\text{and } \therefore S_{m-1} =$$

$$\frac{(l+d)^m - a^m}{md} - (m-1) S_{m-2} \frac{d}{1 \cdot 2} - (m-1)(m-2) S_{m-3} \frac{d^2}{1 \cdot 2 \cdot 3} - \text{\&c.}$$

$$= \frac{(a+nd)^m - a^m}{md} - (m-1) S_{m-2} \frac{d}{1 \cdot 2} - (m-1)(m-2) S_{m-3} \frac{d^2}{1 \cdot 2 \cdot 3} - \text{\&c.}$$

262. Cor. Hence, making $m$ equal to the numbers 1, 2, 3, &c. in order, we derive the following formulæ:

$$S_0 = \frac{(a+nd)^1 - a}{d} = n:$$

$$S_1 = \frac{(a+nd)^2 - a^2}{2d} - S_0 \frac{d}{1.2} = \{2a + (n-1)d\} \frac{n}{1.2}:$$

$$S_2 = \frac{(a+nd)^3 - a^3}{3d} - 2S_1 \frac{d}{1.2} - 2.1\,S_0 \frac{d^2}{1.2.3}$$

$$= \{6a^2 + 6(n-1)ad + (n-1)(2n-1)d^2\} \frac{n}{1.2.3}:$$

$$S_3 = \frac{(a+nd)^4 - a^4}{4d} - 3S_2 \frac{d}{1.2} - 3.2S_1 \frac{d^2}{1.2.3} - 3.2.1\,S_0 \frac{d^3}{1.2.3.4}$$

$$= \{24a^3 + 36(n-1)a^2d + 12(n-1)(2n-1)ad^2 + 6n(n-1)^2d^3\} \frac{n}{1.2.3.4}:$$

&c.........................................................................

If for $m-1$ we substitute $m$ in both members, the general formula becomes somewhat more convenient for practice:

$$S_m = \frac{(a+nd)^{m+1} - a^{m+1}}{(m+1)d} - mS_{m-1} \frac{d}{1.2} - m(m-1)S_{m-2} \frac{d^2}{1.2.3} - \&c.$$

Ex. Let $a = 1 = d$, or the progression become the series of natural numbers 1, 2, 3, &c. $n$; then the formulæ above found, give

$$S_0 = n = 1^0 + 2^0 + 3^0 + \&c. + n^0:$$

$$S_1 = \frac{n(n+1)}{1.2} = 1 + 2 + 3 + \&c. + n:$$

$$S_2 = \frac{n(n+1)(2n+1)}{1.2.3} = 1^2 + 2^2 + 3^2 + \&c. + n^2:$$

$$S_3 = \left\{\frac{n(n+1)}{1.2}\right\}^2 = 1^3 + 2^3 + 3^3 + \&c. + n^3:$$

&c...............................................

From these examples we conclude that the sum of the series

$$1^3 + 2^3 + 3^3 + \&c. + n^3$$

is equal to the square of the sum of the series

$$1 + 2 + 3 + \&c. + n.$$

## II. Geometrical Progression.

263. Def. A *Geometrical Progression* is a series of three or more magnitudes in continued geometrical proportion, and it is characterized by the circumstance of every term having to that which immediately precedes it, the same *Common Ratio*, which may be either integral, fractional or irrational.

Thus, $a, 2a, 4a$, &c.; $a, ab, ab^2$, &c.; $ax, \frac{a^2}{x}, \frac{a^3}{x^3}$, &c.; $x, \frac{\sqrt{x}}{y}, \frac{1}{y^2}$, &c. are geometrical progressions, of which the ratios are $2$, $b$, $\frac{a}{x^2}$ and $\frac{1}{y\sqrt{x}}$ respectively.

264. Cor. Hence it follows that the terms of a geometrical progression taken at equal intervals are likewise in geometrical progression.

265. *In a geometrical progression, given the first term and the common ratio, to find the* n<sup>th</sup> *term and the sum of* n *terms.*

Let $a$ be the first term, $r$ the common ratio, $l$ the $n^{th}$ term and $S$ the sum of $n$ terms:

then will the series itself be $a, ar, ar^2, ar^3$, &c.;

and since $r$ is not found in the first term and its index increases by unity in each term from the second, we shall obviously have

$$l = ar^{n-1}.$$

Also, $S = a + ar + ar^2 + \&c. + ar^{n-2} + ar^{n-1}$

$= a + r(a + ar + \&c. + ar^{n-3} + ar^{n-2})$

$= a + r(S - ar^{n-1}) = a + rS - ar^n$;

$$\therefore \ (r-1)S = a(r^n - 1), \text{ and } S = a\left(\frac{r^n-1}{r-1}\right).$$

If the ratio be a negative quantity $-r$, we shall have

$$l = \pm ar^{n-1}, \text{ and } S = a\left(\frac{\pm r^n + 1}{r+1}\right);$$

and the upper or lower sign is to be used in each of these formulæ according as $n$ is odd or even.

Ex. 1. Required the $n^{\text{th}}$ term and the sum of $n$ terms of the geometric series, 1, 2, 4, 8, &c.

Generally, $l = ar^{n-1}$, and $S = a\left(\frac{r^n-1}{r-1}\right)$; and here $a = 1$, $r = 2$, $\therefore$ the $n^{\text{th}}$ term $= 2^{n-1}$, and the sum of $n$ terms $= 2^n - 1$.

Ex. 2. Find the $n^{\text{th}}$ term and the sum of $n$ terms of the progression, $c^2 - x^2$, $c + x$, $\frac{c+x}{c-x}$, &c.

In this case $a = c^2 - x^2$ and $r = \frac{c+x}{c^2-x^2} = \frac{1}{c-x}$;

$$\therefore \ l = ar^{n-1} = (c^2 - x^2)\frac{1}{(c-x)^{n-1}} = \frac{(c+x)(c-x)}{(c-x)^{n-1}} = \frac{c+x}{(c-x)^{n-2}}:$$

$$\text{also, } S = a\left(\frac{r^n-1}{r-1}\right) = (c^2 - x^2)\left\{\frac{\frac{1}{(c-x)^n} - 1}{\frac{1}{(c-x)} - 1}\right\}$$

$$= \frac{c+x}{(c-x)^{n-2}}\left\{\frac{1-(c-x)^n}{1-(c-x)}\right\}, \text{ or } = \frac{c+x}{(c-x)^{n-2}}\left\{\frac{(c-x)^n - 1}{(c-x) - 1}\right\}.$$

Ex. 3. To find the $n^{\text{th}}$ term and the sum of $n$ terms of the series,

$$\frac{1}{\sqrt{x}} - \frac{\sqrt{b}}{x} + \frac{b}{x\sqrt{x}} - \frac{b\sqrt{b}}{x^2} + \&c.$$

Here $a = \dfrac{1}{\sqrt{x}}$, and $r = -\sqrt{\dfrac{b}{x}}$, $\therefore\ r^{n-1} = \pm \dfrac{b^{\frac{n-1}{2}}}{x^{\frac{n-1}{2}}}$;

whence we obtain $l = \pm \dfrac{1}{\sqrt{x}} \dfrac{b^{\frac{n-1}{2}}}{x^{\frac{n-1}{2}}} = \pm \dfrac{b^{\frac{n-1}{2}}}{x^{\frac{n}{2}}} = \pm \sqrt{\dfrac{b^{n-1}}{x^n}}$;

in which the upper or lower sign must manifestly be used according as $n$ is odd or even:

$$\text{also, } S = \frac{1}{\sqrt{x}}\left\{\frac{\mp \dfrac{b^{\frac{n}{2}}}{x^{\frac{n}{2}}} - 1}{-\dfrac{b^{\frac{1}{2}}}{x^{\frac{1}{2}}} - 1}\right\} = \frac{1}{\sqrt{x}}\left\{\frac{1 \pm \dfrac{b^{\frac{n}{2}}}{x^{\frac{n}{2}}}}{1 + \dfrac{b^{\frac{1}{2}}}{x^{\frac{1}{2}}}}\right\}$$

$$= \frac{1}{x^{\frac{n}{2}}}\left\{\frac{x^{\frac{n}{2}} \pm b^{\frac{n}{2}}}{x^{\frac{1}{2}} + b^{\frac{1}{2}}}\right\}, \text{ or } = \frac{1}{\sqrt{x^n}}\left\{\frac{\sqrt{x^n} \pm \sqrt{b^n}}{\sqrt{x} + \sqrt{b}}\right\};$$

wherein the upper or lower sign is applicable according as $n$ is odd or even.

266. Cor. By means of the two expressions,

$$l = ar^{n-1}, \text{ and } S = a\left(\frac{r^n - 1}{r - 1}\right),$$

above investigated, we arrive immediately at the following results:

(1). $a = \dfrac{l}{r^{n-1}} = \dfrac{(r-1)S}{r^n - 1} = rl - (r-1)S.$

(2). $$l = ar^{n-1} = \frac{(r-1)\,r^{n-1}S}{r^n - 1} = S - \frac{S-a}{r}.$$

(3). $$r = \sqrt[n-1]{\frac{l}{a}} = \frac{S-a}{S-l} = \frac{a-S}{l-S}.$$

(4). $$S = \frac{a(r^n - 1)}{r-1} = \frac{rl - a}{r-1} = \frac{l(r^n - 1)}{r^{n-1}(r-1)} = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

The values of $n$ cannot be exhibited in terms of the rest without the aid of logarithms; but in addition to these, we may easily deduce also the following formulæ:

$$r^n - \left(\frac{S}{a}\right) r + \frac{S}{a} - 1 = 0;$$

$$r^n - \left(\frac{S}{S-l}\right) r^{n-1} + \frac{l}{S-l} = 0;$$

$$\text{and} \quad (S-a)\,a^{\frac{1}{n-1}} = (S-l)\,l^{\frac{1}{n-1}}:$$

from the two former of which $r$ may be found in terms of the rest involved with it, by the solutions of equations of $n$ dimensions.

267. *If the number of terms and the two extremes be known, the series may be determined.*

For, retaining the notation hitherto used, we have

$$l = a\,r^{n-1}, \text{ and } \therefore\ r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}},$$

the common ratio: whence the series will be

$$a,\ (a^{n-2}l)^{\frac{1}{n-1}},\ (a^{n-3}l^2)^{\frac{1}{n-1}},\ \text{\&c.},\ (a^2 l^{n-3})^{\frac{1}{n-1}},\ (a l^{n-2})^{\frac{1}{n-1}},\ l.$$

268. Cor. If $m$ be the number of geometric means between $a$ and $l$, we have $n-1=m+1$; and therefore the means themselves are

$$(a^m l)^{\frac{1}{m+1}},\ (a^{m-1} l^2)^{\frac{1}{m+1}},\ \&c.,\ (a^2 l^{m-1})^{\frac{1}{m+1}},\ (a\, l^m)^{\frac{1}{m+1}}.$$

Ex. Insert three geometric means between 1 and 16.

In this instance $a=1$, $l=16$, and $m=3$; whence the common ratio $=\sqrt[4]{16}=2$, and therefore the means are 2, 4, 8, so that 1, 2, 4, 8, 16 are in geometrical progression.

269. *Given the magnitudes of any two terms whose places are known, to find the series.*

Let the $p^{\text{th}}$ and $q^{\text{th}}$ terms be $P$ and $Q$ respectively; then by (265) we shall have $P=ar^{p-1}$, and $Q=ar^{q-1}$;

$$\text{whence } \frac{P}{Q}=r^{p-q}, \text{ and } \therefore\ r=\left(\frac{P}{Q}\right)^{\frac{1}{p-q}}, \text{ the common ratio:}$$

$$\text{also, } a=\frac{P}{r^{p-1}}=P\left(\frac{Q}{P}\right)^{\frac{p-1}{p-q}}=\left(\frac{Q^{p-1}}{P^{q-1}}\right)^{\frac{1}{p-q}}, \text{ the first term:}$$

wherefore the first term and the common ratio being determined, the series becomes known.

$$\text{The } n^{\text{th}} \text{ term} =\left(\frac{Q^{p-n}}{P^{q-n}}\right)^{\frac{1}{p-q}}; \text{ and the sum of } n \text{ terms}$$

$$=\left(\frac{Q^{p-n}}{P^{q-1}}\right)^{\frac{1}{p-q}}\left\{\frac{P^{\frac{n}{p-q}}-Q^{\frac{n}{p-q}}}{P^{\frac{1}{p-q}}-Q^{\frac{1}{p-q}}}\right\}=\left(\frac{P^{q-n}}{Q^{p-1}}\right)^{\frac{1}{q-p}}\left\{\frac{Q^{\frac{n}{q-p}}-P^{\frac{n}{q-p}}}{Q^{\frac{1}{q-p}}-P^{\frac{1}{q-p}}}\right\}.$$

270. It has been seen in the preceding pages that

$$l=ar^{n-1};$$

and from this we infer that no term, wherein $n$ is finite, can ever become $= 0$: if, however, the ratio be a proper fraction represented by $\frac{1}{R}$, we shall have

$$l = \frac{a}{R^{n-1}},$$

which, by continually increasing $n$, may obviously be made less than any assignable quantity: hence, in this case, the sum of the series will evidently admit of a limit beyond which it can never pass; or, in other words, what is usually termed the sum of the series continued *in infinitum* may be finite, and will be represented by

$$\frac{a\left(\frac{1}{R^n} - 1\right)}{\frac{1}{R} - 1}$$

wherein $n$ is indefinitely great: that is, denoting this limit, or the sum of the series continued *in infinitum* by $\Sigma$, we shall have

$$\Sigma = \frac{-a}{\frac{1}{R} - 1} = \frac{a}{1 - \frac{1}{R}} = \frac{a}{1-r}.$$

Ex. To find the limit of the sum of the geometrical series,

$$\frac{1}{3} - \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \text{\&c.}$$

continued *in infinitum*.

Generally, $\Sigma = \frac{a}{1-r}$, and here $a = \frac{1}{3}$, and $r = -\frac{1}{2}$;

whence we obtain $\Sigma = \frac{\frac{1}{3}}{1 + \frac{1}{2}} = \frac{2}{6+3} = \frac{2}{9}$

271. Cor. From the equation $\Sigma = \frac{a}{1-r}$, above established, we have immediately

$$a = (1-r)\Sigma, \text{ and } r = \frac{\Sigma - a}{\Sigma};$$

and thus, of the three quantities $a$, $r$ and $\Sigma$, each has been expressed in terms of the two others.

Ex. 1. In a geometrical series continued *in infinitum*, it is required to find the limit of the ratio when any term is greater than the sum of all the succeeding terms.

Let the series be $a$, $ar$, $ar^2$, $ar^3$, &c. $ar^{n-1}$, $ar^n$, &c.: then since the condition expressed must hold throughout, we must have

$$ar^{n-1} > ar^n + ar^{n+1} + ar^{n+2} + \&c. \textit{ in infinitum}:$$

$$> ar^n\{1 + r + r^2 + \&c. \textit{ in infinitum}\}:$$

$$> \frac{ar^n}{1-r}:$$

whence it is manifest that $1 - r > r$, and $\therefore\ r < \frac{1}{2}$:

that is, any infinite geometrical progression, whose ratio is less than $\frac{1}{2}$, will have any one term greater than the sum of all that follow it.

Hence also, if the ratio $r = \frac{1}{2}$, or the infinite series be

$$a, \frac{a}{2}, \frac{a}{2^2}, \frac{a}{2^3}, \&c. \frac{a}{2^{n-1}}, \frac{a}{2^n}, \&c.,$$

every term will be equal to the sum of those which succeed it.

Ex. 2. There are two infinite geometrical progressions, each beginning from 1, whose sums are $\sigma_1$ and $\sigma_2$; it is required to prove that the sum of the series formed by multiplying together their corresponding terms, is $\dfrac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2 - 1}$.

Let $r_1$ and $r_2$ be the common ratios of the two progressions;

$$\therefore \; \sigma_1 = 1 + r_1 + r_1^2 + \&\text{c. } \textit{in infinitum} = \frac{1}{1 - r_1}, \qquad (1),$$

$$\sigma_2 = 1 + r_2 + r_2^2 + \&\text{c. } \textit{in infinitum} = \frac{1}{1 - r_2}, \qquad (2):$$

and the sum of the series resulting from the multiplication of the corresponding terms of these two will manifestly be

$$1 + (r_1 r_2) + (r_1 r_2)^2 + \&\text{c. } \textit{in infinitum}$$

$$= \frac{1}{1 - r_1 r_2} = \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2 - 1}:$$

by substituting for $r_1$ and $r_2$ their values determined from (1) and (2) respectively.

Ex. 3. The expansion of $(1 + x)^m$ is of intermediate magnitude to $\dfrac{1}{1 - mx}$ and $\dfrac{1}{1 + x}$, whenever $mx$ and $x$ are both proper fractions.

For, it has been proved in (199) that the series

$$1 + mx + m^2 x^2 + m^3 x^3 + \&\text{c. } \textit{in infinitum}$$

is greater than the expansion of $(1 + x)^m$, and the series

$$1 - x + x^2 - x^3 + \&\text{c. } \textit{in infinitum}$$

less than the same quantity:

but by (270) the sums of these infinite series being

$$\frac{1}{1 - mx} \text{ and } \frac{1}{1 + x}$$

respectively, it follows that the expansion of $(1+x)^m$ is, in this case, intermediate in magnitude to

$$\frac{1}{1-mx} \text{ and } \frac{1}{1+x}.$$

Hence also, the expansion of $(1+x)^m$ differs from $\dfrac{1}{1-mx}$ by a quantity less than $\dfrac{1}{1-mx}-\dfrac{1}{1+x}$ or $\dfrac{(m+1)x}{(1-mx)(1+x)}$.

272. *To find the sum of a series of quantities in geometrical progression having their coefficients in arithmetical progression.*

Let the proposed series of magnitudes be

$a$, $(a+b)r$, $(a+2b)r^2$, $(a+3b)r^3$, &c. $\{a+(n-1)b\}r^{n-1}$,

and assume

$$a+(a+b)r+\text{\&c.}+\{a+(n-2)b\}r^{n-2}+\{a+(n-1)b\}r^{n-1}=S,$$

$$\therefore\ ar+(a+b)r^2+\text{\&c.}+\{a+(n-2)b\}r^{n-1}+\{a+(n-1)b\}r^n=rS;$$

$\therefore$ by subtraction, we obtain

$$a+br+br^2+\text{\&c.}+br^{n-1}-\{a+(n-1)b\}r^n=-(r-1)S,$$

$$\text{or } a-\{a+(n-1)b\}r^n+br\left(\frac{r^{n-1}-1}{r-1}\right)=-(r-1)S:$$

$$\text{whence } S=\frac{\{a+(n-1)b\}r^n-a-br\left(\dfrac{r^{n-1}-1}{r-1}\right)}{r-1}$$

$$=\frac{\{a+(n-1)b\}r^n-a}{r-1}-\frac{br(r^{n-1}-1)}{(r-1)^2}.$$

273. Cor. If the ratio of the geometrical progression be a proper fraction, and the number of terms be supposed indefinitely great, we shall have, corresponding thereto,

$$\Sigma = \frac{a-(a-b)r}{(1-r)^2}.$$

Ex. Required the sums of the series $1+2x+3x^2+$ &c. to $n$ terms, and *in infinitum*.

Let $1+2x+3x^2+\text{\&c.}+(n-1)x^{n-2}+nx^{n-1}=S$;

$$\therefore\ x+2x^2+\text{\&c.}+(n-2)x^{n-1}+(n-1)x^{n-1}+nx^n=xS:$$

whence $1+x+x^2+\text{\&c.}+x^{n-1}-nx^n=S(1-x)$,

$$\text{and } \therefore\ S=\frac{nx^n}{x-1}-\frac{x^n-1}{(x-1)^2}:$$

which, when $x$ is a proper fraction and the series is indefinite, gives

$$\Sigma=\frac{1}{(1-x)^2}.$$

274. *To find the sum of a series of fractions, whose numerators are in arithmetical, and denominators in geometrical, progression.*

Let us assume the required sum,

$$\frac{a}{1}+\frac{a+b}{r}+\frac{a+2b}{r^2}+\text{\&c.}+\frac{a+(n-2)b}{r^{n-2}}+\frac{a+(n-1)b}{r^{n-1}}=S:$$

$$\text{then } \frac{a}{r}+\frac{a+b}{r^2}+\text{\&c.}+\frac{a+(n-2)b}{r^{n-1}}+\frac{a+(n-1)b}{r^n}=\frac{1}{r}S;$$

$$\text{whence } a+\frac{b}{r}+\frac{b}{r^2}+\text{\&c.}+\frac{b}{r^{n-1}}-\frac{a+(n-1)b}{r^n}=\frac{r-1}{r}S:$$

$$\therefore\ a+\frac{b(r^{n-1}-1)}{r^{n-1}(r-1)}-\frac{a+(n-1)b}{r^n}=\frac{r-1}{r}S:$$

$$\text{and thence } S=\frac{ra}{r-1}-\frac{a+(n-1)b}{r^{n-1}(r-1)}+\frac{b(r^{n-1}-1)}{r^{n-2}(r-1)^2}.$$

275. Cor. Wherefore, if $r$ be greater than 1 and the number of terms be indefinitely increased, we shall obtain

$$\Sigma = \frac{ar}{r-1}.$$

Ex. Let it be required to find the sums of the series

$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} +$ &c. to $n$ terms and *in infinitum*.

Let $1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \text{\&c.} + (n-1)\frac{1}{2^{n-1}} + n\frac{1}{2^n} = S,$

$$\therefore\ 1 \cdot \frac{1}{2^2} + 2 \cdot \frac{1}{2^3} + \text{\&c.} + (n-2)\frac{1}{2^{n-1}} + (n-1)\frac{1}{2^n} + n\frac{1}{2^{n+1}} = \frac{1}{2}S;$$

whence, by subtraction, $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{\&c.} + \frac{1}{2^n} - \frac{n}{2^{n+1}} = \frac{1}{2}S:$

$$\text{or } 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}} = \frac{1}{2}S;$$

$$\therefore\ S = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}:$$

and if $n$ be supposed to become indefinitely great, we find

$$\Sigma = 2.$$

## III. Harmonical Progression.

276. Def. An *Harmonical Progression* is a series of magnitudes in continued harmonical proportion; or such that if any three consecutive terms be taken, the first is to the third, as the difference between the first and second is to the difference between the second and third, as appears from the definition expressed in (248).

Thus, if $a$, $b$, $c$, $d$, &c. be the consecutive terms of an harmonical progression, we shall have

$$a : c :: a-b : b-c;$$

$$b : d :: b-c : c-d, \text{ \&c.};$$

and the characteristic property of this series is contained in the following article.

277. *The reciprocals of the terms of an harmonical progression are in arithmetical progression.*

As above, let $a$, $b$, $c$, $d$, $e$, &c. be the terms of the series;

$$\text{then } a : c :: a-b : b-c,$$

$$\text{and } \therefore \text{ by (238), } ab-ac=ac-bc;$$

$$\text{wherefore } \frac{ab}{abc}-\frac{ac}{abc}=\frac{ac}{abc}-\frac{bc}{abc},$$

$$\text{or } \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}:$$

$$\text{similarly } \frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b}; \quad \frac{1}{e}-\frac{1}{d}=\frac{1}{d}-\frac{1}{c}; \text{ \&c.:}$$

whence it is obvious that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$, &c. form an arithmetical progression.

278. Cor. 1. From the last article it appears that the terms of an harmonical progression, taken at equal distances, are also in harmonical progression.

279. Cor. 2. If $x$ and $y$ be any two adjacent terms of the harmonical series $a$, $b$, $c$, $d$, &c. $x$, $y$, &c., then will

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{y}-\frac{1}{x}, \text{ or } \frac{a-b}{ab}=\frac{x-y}{xy};$$

$$\therefore ab : xy :: a-b : x-y:$$

or the product of the first two terms is to the product of any two adjacent terms, as the difference between the first two is to the difference between the other two.

280. *Given the first two terms of an harmonical progression, to find the* n$^{\text{th}}$ *term.*

Let $a$ and $b$ be the first two terms, $l$ the $n^{\text{th}}$ term required: then it is obvious that $\frac{1}{l}$ is the $n^{\text{th}}$ term of an arithmetical progression, the first two terms of which are $\frac{1}{a}$ and $\frac{1}{b}$, and therefore we have the common difference $=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$:

$$\therefore \text{ by } (256),\ \frac{1}{l}=\frac{1}{a}+\frac{(n-1)(a-b)}{ab}=\frac{(n-1)a-(n-2)b}{ab};$$

$$\therefore\ l=\frac{ab}{(n-1)a-(n-2)b}.$$

The sum of $n$ terms of an harmonical progression cannot be exhibited generally in finite terms.

281. Cor. 1. Making $n$ equal to the numbers 1, 2, 3, 4, &c. in succession, we shall have the corresponding terms of the series equivalent to $a$, $b$, $\frac{ab}{2a-b}$, $\frac{ab}{3a-2b}$, &c.

The series may also be continued backwards by a similar formula.

Ex. If the first two terms be 4 and 3, find the series.

Here we have $a=4$, and $b=3$; and the formula above gives

$$\text{the } n^{\text{th}} \text{ term} = \frac{12}{(n-1)4-(n-2)3}=\frac{12}{n+2}:$$

whence the harmonical series is 4, 3, $\frac{12}{5}$, 2, $\frac{12}{7}$, &c.

282. Cor. 2. From the formula above investigated,

$$l = \frac{ab}{(n-1)a-(n-2)b},$$

we derive immediately the following results:

$$a = \frac{(n-2)bl}{(n-1)l-b}, \quad b = \frac{(n-1)al}{(n-2)l+a} \quad \text{and} \quad n = \frac{ab+(a-2b)l}{(a-b)l}.$$

283. *If the two extremes and the number of terms be known, the intervening terms may be found.*

For, $d$ the common difference of the reciprocals of the terms is easily found $= \frac{a-l}{(n-1)al}$; whence the arithmetical progression will be

$$\frac{1}{a}, \frac{a+(n-2)l}{(n-1)al}, \frac{2a+(n-3)l}{(n-1)al}, \text{\&c.}, \frac{(n-3)a+2l}{(n-1)al}, \frac{(n-2)a+l}{(n-1)al}, \frac{1}{l}:$$

and consequently the reciprocals of these, or the harmonical series, will be

$$a, \frac{(n-1)al}{a+(n-2)l}, \frac{(n-1)al}{2a+(n-3)l}, \text{\&c.}, \frac{(n-1)al}{(n-3)a+2l}, \frac{(n-1)al}{(n-2)a+l}, l.$$

284. Cor. From this proposition $m$ harmonical means between $a$ and $l$ may easily be found.

For, since $n=m+2$, we have $n-1=m+1$; and therefore the $m$ harmonical means are

$$\frac{(m+1)al}{a+ml}, \frac{(m+1)al}{2a+(m-1)l}, \text{\&c.} \frac{(m+1)al}{(m-1)a+2l}, \frac{(m+1)al}{ma+l}.$$

Ex. Insert two harmonic means between 3 and 12.

Here $a=3$, $l=12$ and $m=2$; whence the means will be

$$\frac{3.3.12}{3+2.12} = 4, \quad \text{and} \quad \frac{3.3.12}{2.3+1.12} = 6;$$

and it is easily seen that 3, 4, 6 and 12 are in harmonical progression.

285. *Given the magnitudes of any two terms in known situations, to find the harmonical progression.*

Let $P$ and $Q$ represent the $p^{\text{th}}$ and $q^{\text{th}}$ terms; then, as in (260), if $d$ be the common difference of the reciprocals of the terms, and $a$ the reciprocal of the first term, we shall have

$$\frac{1}{P} = a + (p-1)d, \quad \text{and} \quad \frac{1}{Q} = a + (q-1)d:$$

$$\text{whence } \frac{1}{P} - \frac{1}{Q} = (p-q)d, \quad \text{and } \therefore \ d = -\frac{(P-Q)}{(p-q)PQ}:$$

$$\therefore \ a = \frac{1}{P} + \frac{(p-1)(P-Q)}{(p-q)PQ} = \frac{(p-1)P-(q-1)Q}{(p-q)PQ};$$

so that both the first term and the common difference of their reciprocals being found, the terms of the series themselves become known.

286. Though the sum of a series of quantities in harmonical progression cannot be generally determined, there are some cases in which a good approximation may be found.

Thus, in the series $\dfrac{1}{a+b}, \dfrac{1}{a+2b}, \dfrac{1}{a+3b}$, &c. wherein $b$ is very small compared with $a$, we have by (236),

$$a+2b = \frac{(a+b)^2}{a}, \quad a+3b = \frac{(a+b)^3}{a^2}, \text{ \&c. nearly;}$$

therefore an approximate value of the sum of the harmonical series,

$$\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \text{\&c. to } n \text{ terms, is}$$

$$\frac{1}{a+b}+\frac{a}{(a+b)^2}+\frac{a^2}{(a+b)^3}+\text{\&c. to } n \text{ terms},$$

$$\text{which} = \frac{(a+b)^n - a^n}{b\,(a+b)^n}.$$

From the preceding operation, it appears that quantities in arithmetical progression whose differences are very small compared to themselves, may *approximately* be considered to be in geometrical progression.

287. If three equidistant terms of an harmonical progression be $\dfrac{1}{a+(n-1)d}$, $\dfrac{1}{a}$, $\dfrac{1}{a-(n-1)d}$, we have

$$\frac{1}{a-(n-1)d}+\frac{1}{a+(n-1)d}=\frac{2a}{a^2-(n-1)^2d^2},$$

which is manifestly greater than $\dfrac{2}{a}$: that is, the sum of any two terms of an harmonical series is greater than twice the intermediate mean term; and it is evident that this excess is the greater as they are more remote from it.

Whence we shall obviously have the sum of $n$ terms of the series greater than $n$ times the middle term; and therefore by the continued increase of $n$, the sum of every harmonical series may be made greater than any quantity that can be assigned.

Hence the sum of the reciprocals of the natural numbers, or

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{\&c. continued } \textit{in infinitum},$$

is indefinitely great.

288. The propositions proved, and formulæ investigated in this Chapter, will be further illustrated and applied in the following set of miscellaneous theorems and problems.

(1). If 100 stones be placed in a right line, exactly a yard asunder, the first being one yard from a basket, what distance will a person go who gathers them up singly, returning with each to the basket?

Since he goes 2 yards for the first stone, 4 for the second, 6 for the third, &c. and 200 for the last, from the basket and back again: the distances travelled form an arithmetical progression whose first term = 2, last term = 200, and number of terms = 100:

$$\therefore \text{ the whole space } = (a+l)\frac{n}{2} = (2+200)\,50$$

$$= 10100 \text{ yards} = 5 \text{ miles } 1300 \text{ yards.}$$

(2). Of four quantities in arithmetical progression, the sum of the squares of the extremes $= a^2$, and the sum of the squares of the means $= b^2$: find them.

Here $x-3y$, $x-y$, $x+y$ and $x+3y$, which are in arithmetical progression, having the common difference $2y$, may represent the quantities required: therefore by the question, we have

$$(x-3y)^2+(x+3y)^2=a^2, \text{ or } 2x^2+18y^2=a^2:$$

$$(x-y)^2+(x+y)^2=b^2, \text{ or } 2x^2+2y^2=b^2:$$

$$\text{whence } 16y^2=a^2-b^2 \text{ and } y=\pm\frac{\sqrt{a^2-b^2}}{4}:$$

$$\text{and } \therefore\ 2x^2=b^2-2y^2=b^2-\frac{a^2-b^2}{8}=\frac{9b^2-a^2}{8};$$

$$\therefore\ x=\pm\frac{\sqrt{9b^2-a^2}}{4};$$

and the quantities required are

$$\frac{\sqrt{9b^2-a^2}-3\sqrt{a^2-b^2}}{4}, \quad \frac{\sqrt{9b^2-a^2}-\sqrt{a^2-b^2}}{4},$$

$$\frac{\sqrt{9b^2-a^2}+\sqrt{a^2-b^2}}{4} \text{ and } \frac{\sqrt{9b^2-a^2}+3\sqrt{a^2-b^2}}{4}.$$

(3). Given four magnitudes $a$, $b$, $c$ and $d$, to find another which being added to each of them, the squares of the sums shall be in arithmetical progression.

Let $x$ be the required magnitude; then by the question

$$(a+x)^2-(b+x)^2=(c+x)^2-(d+x)^2,$$

which gives immediately $x = \dfrac{b^2+c^2-a^2-d^2}{2(a-b-c+d)}$;

the required magnitudes, which are easily proved to be in arithmetical progression, will be

$$\frac{(a-b)^2+(a-c)^2-(a-d)^2}{2(a-b-c+d)}, \quad \frac{(b-a)^2+(b-d)^2-(b-c)^2}{2(a-b-c+d)},$$

$$\frac{(c-a)^2+(c-d)^2-(c-b)^2}{2(a-b-c+d)} \text{ and } \frac{(d-b)^2+(d-c)^2-(d-a)^2}{2(a-b-c+d)}.$$

Similarly, when the cubes are in arithmetical progression.

(4). To find four magnitudes in arithmetical progression, the product of the means being $a^2$, and that of the extremes $b^2$.

Let $2x$ denote the sums of the means; then will their semi-difference $=\sqrt{x^2-a^2}$:

$\therefore$ the means are $x+\sqrt{x^2-a^2}$ and $x-\sqrt{x^2-a^2}$,

and thence the extremes are $x+3\sqrt{x^2-a^2}$ and $x-3\sqrt{x^2-a^2}$:

$\therefore$ $x^2-9(x^2-a^2)=b^2$, by the question;

from which $x^2=\dfrac{9a^2-b^2}{8}$, and $x=\pm\frac{1}{2}\sqrt{\dfrac{9a^2-b^2}{2}}$:

therefore by substitution, we shall readily find that

$$\text{the means are } \frac{1}{2}\sqrt{\frac{9a^2-b^2}{2}}+\frac{1}{2}\sqrt{\frac{a^2-b^2}{2}},$$

$$\text{and } \frac{1}{2}\sqrt{\frac{9a^2-b^2}{2}}-\frac{1}{2}\sqrt{\frac{a^2-b^2}{2}}:$$

$$\text{and the extremes are } \frac{1}{2}\sqrt{\frac{9a^2-b^2}{2}}+\frac{3}{2}\sqrt{\frac{a^2-b^2}{2}},$$

$$\text{and } \frac{1}{2}\sqrt{\frac{9a^2-b^2}{2}}-\frac{3}{2}\sqrt{\frac{a^2-b^2}{2}}.$$

(5). Given the sum of $n$ terms of an arithmetical progression $= s$, and the sum of their squares $= S$; to find the series.

First, let $n$ be odd and the terms of the series be expressed by $x$, $x \pm y$, $x \pm 2y$, &c.; then we have

$$nx = s, \text{ and } \therefore\ x = \frac{s}{n}:$$

$$\text{also, } nx^2 + 2y^2\left\{1 + 2^2 + 3^2 + \text{\&c. to } \tfrac{1}{2}(n-1) \text{ terms}\right\} = S:$$

$$\text{but by (261), } 1^2+2^2+3^2+\text{\&c. to } \tfrac{1}{2}(n-1) \text{ terms} = \frac{(n-1)\,n\,(n+1)}{1.2.3.4}:$$

$$\therefore \frac{s^2}{n} + \frac{(n-1)\,n\,(n+1)}{1.3.4}\,y^2 = S, \text{ and } y = \pm\frac{2}{n}\sqrt{\frac{3(nS-s^2)}{n^2-1}}:$$

and thus the series is found.

Again, let $n$ be even and the terms of the required series be denoted by

$$x \pm y, \quad x \pm 3y, \quad x \pm 5y, \quad \text{\&c.};$$

then we have $nx = s$ and $\therefore\ x = \frac{s}{n}$, as before:

also, $nx^2 + 2y^2\{1^2 + 3^2 + 5^2 + \&c. \text{ to } \frac{1}{2}n \text{ terms}\} = S$:

but by (261), $1^2 + 3^2 + 5^2 + \&c. \text{ to } \frac{1}{2}n \text{ terms} = \frac{(n-1)\,n\,(n+1)}{1\,.\,2\,.\,3}$:

$$\therefore\ \frac{s^2}{n} + \frac{(n-1)\,n\,(n+1)}{1\,.\,3}\,y^2 = S, \text{ and } y = \pm\frac{1}{n}\sqrt{\frac{3\,(nS - s^2)}{n^2 - 1}};$$

whence the series is found.

(6). Find three magnitudes in geometrical progression whose continued product is $a^3$, and the sum of their cubes $b^3$.

Let $x$, $xy$ and $xy^2$ be the quantities required:

then $x^3y^3 = a^3$ and $x^3 + x^3y^3 + x^3y^6 = b^3$:

also, from the first of these, $y^3 = \frac{a^3}{x^3}$:

$\therefore$ by substitution in the second, we obtain

$$x^3 + a^3 + \frac{a^6}{x^3} = b^3, \text{ and } x^6 + (a^3 - b^3)\,x^3 + a^6 = 0:$$

whence $x$ may be found by completing the square, and thence $y$, and the quantities sought.

(7). Of four magnitudes in geometrical progression, given the sum of the two least $= a$ and that of the two greatest $= b$; to find them.

Let $x$ denote the first, and $y$ the third magnitude:

then $a - x =$ the second, and $b - y =$ the fourth:

$$\therefore\ x : a - x :: y : b - y,$$

$$\text{or } bx - xy = ay - xy, \text{ whence } bx = ay:$$

$$\text{also } xy = (a-x)^2:$$

$$\text{whence we have } \frac{bx^2}{a} = (a-x)^2, \text{ and } x\sqrt{\frac{b}{a}} = a - x:$$

$$\therefore\ x = \frac{a\sqrt{a}}{\sqrt{a}+\sqrt{b}} \text{ and } y = \frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}};$$

$$\text{wherefore } a - x = \frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}} \text{ and } b - y = \frac{b\sqrt{b}}{\sqrt{a}+\sqrt{b}}.$$

(8). Find six magnitudes in geometrical progression, whose sum shall be $a$, and the sum of the two extremes shall be $b$.

Let $x$ be the first term, and $y$ the common ratio:

$$\text{then } x + xy + xy^2 + xy^3 + xy^4 + xy^5 = a,$$

$$\text{and } x + xy^5 = b, \text{ by the question}:$$

$$\text{that is, } \frac{x(y^6-1)}{y-1} = a \text{ and } x(y^5+1) = b:$$

$$\therefore\ \frac{b}{y^5+1}\,\frac{y^6-1}{y-1} = a, \text{ and } b(y^4+y^2+1) = a(y^4-y^3+y^2-y+1):$$

$$\therefore\ (a-b)(y^4+y^2+1) = ay(y^2+1):$$

$$\therefore\ (y^2+1)^2 - \frac{ay}{a-b}(y^2+1) - y^2 = 0:$$

$$\therefore\ (y^2+1)^2 - \frac{ay}{a-b}(y^2+1) + \frac{a^2y^2}{4(a-b)^2} = \left\{\frac{a^2}{4(a-b)^2}+1\right\}y^2:$$

$$\text{whence } y^2 + 1 - \frac{ay}{2(a-b)} = \pm y\sqrt{\frac{a^2}{4(a-b)^2}+1}:$$

and by the solution of this quadratic equation $y$ may be determined, and then $x$ from the equation $x = \dfrac{b}{y^5+1}$.

(9). To find five quantities in geometrical progression, when the sum of the even terms $= 2a$, and the sum of the odd terms $= 2b$.

Taking $x$ for the third or mean term, we have immediately the square of half the difference of the even terms $= a^2 - x^2$:

$\therefore$ the even terms are $a + \sqrt{a^2 - x^2}$ and $a - \sqrt{a^2 - x^2}$,

and the extremes $\dfrac{(a + \sqrt{a^2 - x^2})^2}{x}$ and $\dfrac{(a - \sqrt{a^2 - x^2})^2}{x}$:

whence the sum of the extremes $= \dfrac{4a^2 - 2x^2}{x}$,

and the sum of the odd terms $= \dfrac{4a^2 - x^2}{x}$:

$\therefore$ by the question, $\dfrac{4a^2 - x^2}{x} = 2b$;

from which $x = -b \pm \sqrt{b^2 + 4a^2}$, and thus the terms may be found.

(10). To find four magnitudes in geometrical progression, having given the sum of the means $= 2a$, and the sum of the extremes $= 2b$.

Let $x$ denote the product of the means which is equal to the product of the extremes:

then the semi-difference of the means $= \sqrt{a^2 - x}$;

$\therefore$ the means are $a + \sqrt{a^2 - x}$ and $a - \sqrt{a^2 - x}$;

and the extremes are $\dfrac{(a + \sqrt{a^2 - x})^3}{x}$ and $\dfrac{(a - \sqrt{a^2 - x})^3}{x}$:

whence, by the question, is obtained

$$\frac{8a^3 - 6ax}{x} = 2b, \text{ and } \therefore\ x = \frac{4a^3}{3a+b}:$$

and thus the quantities required are found.

(11). Given the sum of four quantities in geometrical progression $= 4a$, and the sum of their squares $= 16b^2$: to find them.

Let the sum of the means $= 2x$,

$\therefore$ the sum of the extremes $= 4a - 2x$:

then the semi-difference of the means $= x\sqrt{\dfrac{a-x}{a+x}}$, by the last:

$\therefore$ the means are $x\left(1 + \sqrt{\dfrac{a-x}{a+x}}\right)$ and $x\left(1 - \sqrt{\dfrac{a-x}{a+x}}\right)$,

the extremes $\dfrac{a+x}{2}\left(1 + \sqrt{\dfrac{a-x}{a+x}}\right)^3$ and $\dfrac{a+x}{2}\left(1 - \sqrt{\dfrac{a-x}{a+x}}\right)^3$:

whence we have, by the second condition of the question,

$$\frac{8a}{a+x}(2a^2 - x^2) = 16b^2,$$

$$\text{from which } x = \frac{-b^2 \pm \sqrt{a^4 + (a^2 - b^2)^2}}{a},$$

and thus the quantities are discovered.

(12). To find four quantities in harmonical progression, so that their sum may $= a$, and the sum of their reciprocals $= b$.

Let $\frac{1}{x-3y}$, $\frac{1}{x-y}$, $\frac{1}{x+y}$ and $\frac{1}{x+3y}$ which are in harmonical progression, represent the quantities sought:

$$\text{then } \frac{1}{x-3y}+\frac{1}{x-y}+\frac{1}{x+y}+\frac{1}{x+3y}=a,$$

$$\text{and } x-3y+x-y+x+y+x+3y=b:$$

from the second $x=\frac{b}{4}$; and $\therefore$ from the first by substitution, we obtain

$$\frac{b}{2}\left\{\frac{b^2}{8}-10y^2\right\}=a\left(\frac{b^2}{16}-9y^2\right)\left(\frac{b^2}{16}-y^2\right),$$

from which $y$ may be found by the solution of a quadratic.

(13). *A* gives to *B* as many counters as he has already, and *B* returns to *A* as many as he had then kept, and they afterwards find their numbers to be $a$ and $b$ respectively: what number had each at first?

Let $x$ denote *A*'s original number,

then $a+b-x$ will be *B*'s number at first;

$\therefore x-(a+b-x)=2x-(a+b)=$ the number *A* had left after his present to *B*, and $a+b-x+a+b-x=2(a+b-x)=$ *B*'s number then:

$\therefore 2x-(a+b)+2x-(a+b)=4x-2(a+b)=$ the number *A* had after *B*'s present:

whence $4x-2(a+b)=a$, by the question:

$$\therefore x=\frac{3a+2b}{4} \text{ and } a+b-x=\frac{a+2b}{4},$$

which are the representatives of the original numbers.

Now if the same steps be repeated a second time, it is obvious that we have merely to substitute in the places of $a$ and $b$, the quantities $\frac{3a+2b}{4}$ and $\frac{a+2b}{4}$ respectively;

$$\text{and thus we find } A\text{'s original number} = \frac{11a+10b}{16},$$

$$\ldots\ldots\ldots\ldots B\text{'s} \ldots\ldots\ldots\ldots = \frac{5a+6b}{16}:$$

also, for a third time, we must substitute

$\frac{11a+10b}{16}$ and $\frac{5a+6b}{16}$ in the places of $a$ and $b$ respectively;

$$\text{and then we have } A\text{'s number at first} = \frac{43a+42b}{64},$$

$$\ldots\ldots\ldots\ldots B\text{'s} \ldots\ldots\ldots\ldots = \frac{21a+22b}{64}:$$

so again, for a fourth time, we get

$$A\text{'s original number} = \frac{171a+170b}{256},$$

$$B\text{'s} \ldots\ldots\ldots\ldots = \frac{85a+86b}{256}; \text{ and so on:}$$

but in the numerators of the original numbers of $A$, we observe that the coefficients 3, 11, 43, 171, &c. of $a$ are so connected that each of them is less by unity than four times that which immediately precedes it, and that the coefficients of $b$ are the doubles of the numbers 1, 5, 21, 85, &c.: also,

$$3 = 4^1 - 1, \qquad 1 = 1,$$

$$11 = 4^2 - 4 - 1, \qquad 5 = 4^1 + 1,$$

$$43 = 4^3 - 4^2 - 4 - 1, \qquad 21 = 4^2 + 4 + 1,$$

$$171 = 4^4 - 4^3 - 4^2 - 4 - 1; \qquad 85 = 4^3 + 4^2 + 4 + 1;$$

&c. ...........................................

whence we are led by analogy to conclude generally that for $n$ such processes as are mentioned in the question,

$$\text{the coefficient of } a = 4^n - (4^{n-1} + 4^{n-2} + \&c. + 1) = \frac{2.4^n + 1}{3},$$

$$\text{and that of } b = 2(4^{n-1} + 4^{n-2} + \&c. + 1) = \frac{2(4^n - 1)}{3}:$$

from which we determine the original numbers of $A$ and $B$ to be respectively

$$\frac{2.4^n + 1}{3.4^n} a + \frac{2(4^n - 1)}{3.4^n} b,$$

$$\text{and } \frac{4^n - 1}{3.4^n} a + \frac{4^n + 2}{3.4^n} b.$$

These results may be verified by induction; for, in the two original numbers represented by

$$\frac{3a + 2b}{4} \text{ and } \frac{a + 2b}{4},$$

if we substitute these latter expressions for $a$ and $b$, we shall in the original number of $A$ have

$$\text{the coefficient of } a = \frac{3(2.4^n + 1) + 2(4^n - 1)}{3.4^n} = \frac{2.4^{n+1} + 1}{3.4^n},$$

$$\text{and that of } b = \frac{6(4^n - 1) + 2(4^n + 2)}{3.4^n} = \frac{2(4^{n+1} - 1)}{3.4^n};$$

therefore having performed the specified operation $n+1$ times,

$$\text{the original number of } A = \frac{2.4^{n+1} + 1}{3.4^{n+1}} a + \frac{2(4^{n+1} - 1)}{3.4^{n+1}} b,$$

$$\text{and the original number of } B = \frac{4^{n+1} - 1}{3.4^{n+1}} a + \frac{4^{n+1} + 2}{3.4^{n+1}} b;$$

as they ought to be.

# CHAP. X.

## *On Variations, Permutations and Combinations.*

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### I. Variations and Permutations.

289. Def. The *Variations* and *Permutations* of any number of quantities or things are the different orders which can be formed out of them with regard to position, when a certain number of them and the whole are respectively taken at a time.

Thus, of the three things represented by $a$, $b$ and $c$, the variations formed by taking one at a time are

$$a,\ b,\ c:$$

and, when taken two and two together, the variations are

$$ab,\ ba,\ ac,\ ca,\ bc,\ cb:$$

whereas, the permutations formed by taking them all together will be

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

Without attending to the distinction above noticed, it is customary to make use of the terms, *Permutations*, *Variations*, *Alternations* and *Changes*, promiscuously, whether the whole or part be taken at a time: but we shall at present adhere to the definition just laid down.

290. Before investigating the formulæ peculiar to this subject, we shall endeavour to point out how the different variations of any number of things may be practically exhibited, and how the number of them may be obtained.

If we have the $m$ things $a$, $b$, $c$, $d$, &c, $l$, it is evident that the variations of them taken *one* at a time are

$$a, \ b, \ c, \ d, \ \&c, \ l;$$

the number of which is therefore $m$.

Of the $m-1$ things $b$, $c$, $d$, &c, $l$, it is manifest that there will be $m-1$ variations taken one at a time: wherefore if $a$ be placed before each of these, we shall have the variations

$$ab, \ ac, \ ad, \ \&c, \ al;$$

*two* being taken at a time, in which $a$ stands first and whose number is therefore $m-1$: this may be written

$$a(b+c+d+\&c, +l);$$

and we shall similarly have $m-1$ variations in which each of the quantities $b$, $c$, $d$, &c, $l$ stands first, represented by

$$\begin{aligned}
&b(a+c+d+\&c.+l),\\
&c(a+b+d+\&c.+l),\\
&d(a+b+c+\&c.+l),\\
&\&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&l(a+b+c+\&c.+k):
\end{aligned}$$

wherefore, upon the whole, there are $m(m-1)$ variations of the $m$ quantities taken *two* together.

Again, of the $m-1$ things $b$, $c$, $d$, &c, $l$, there may be formed, by taking two together, $(m-1)(m-2)$ variations, as appears above, and which may be represented by

$$\begin{aligned}
&b(c+d+\&c.+l),\\
&c(b+d+\&c.+l),\\
&d(b+c+\&c.+l),\\
&\&c.\ldots\ldots\ldots\ldots\ldots\ldots\\
&l(b+c+\&c.+k):
\end{aligned}$$

wherefore if $a$ be placed before each of these, we shall obviously have the $(m-1)(m-2)$ variations

$$\begin{aligned}&ab\,(c+d+\&c.+l),\\&ac\,(b+d+\&c.+l),\\&ad\,(b+c+\&c.+l),\\&\&c.\dots\dots\dots\dots\\&al\,(b+c+\&c.+k):\end{aligned}$$

formed by taking *three* of the quantities together, where $a$ stands first; and the same being true of $b$, $c$, $d$, &c. $l$, it follows that the total number of variations formed by taking *three* of the quantities at a time will be $m(m-1)(m-2)$.

By pursuing the same mode of reasoning we shall be enabled easily to exhibit the variations arising from taking four, five, &c. quantities at a time, and whose numbers may be similarly proved to be

$$\begin{aligned}&m(m-1)(m-2)(m-3),\\&m(m-1)(m-2)(m-3)(m-4),\ \&c.\end{aligned}$$

respectively.

The results already obtained, united with the method of induction, would lead immediately to the number of variations formed by taking any number of the quantities together; but the formula for that purpose will be more conveniently deduced in the following proposition.

291. *To find the number of Variations that can be formed out of* m *different things, when* r *of them are always taken together.*

Let $V_r$ be the number of variations of $m$ things $a$, $b$, $c$, &c. taken $r$ together;

$V_{r-1}$ .............................. $m-1$ ............... $r-1$ ......

$V_{r-2}$ .............................. $m-2$ ............... $r-2$ ......

&c. .............................. &c. ............... &c. ........;

then, if to each of the variations of $m-1$ things, as $b$, $c$, &c. formed by taking $r-1$ at a time, one of the things as $a$ be prefixed, we shall manifestly have $V_{r-1}$ variations of $m$ things taken $r$ together, in which $a$ stands first: and similarly of $b$, $c$, &c.; therefore, upon the whole, there will be $m\,V_{r-1}$ variations of $m$ things taken $r$ together: that is,

$$V_r = m\,V_{r-1}:$$

and by a similar mode of reasoning we shall manifestly have

$$V_{r-1} = (m-1)\,V_{r-2};$$
$$V_{r-2} = (m-2)\,V_{r-3}; \text{ \&c.}$$

and so on; until we arrive at $V_1$ which is manifestly the number of variations of $m-r+1$ things taken one at a time, and is therefore $= m-r+1$: whence we have

$$V_r = m(m-1)(m-2).\,\&\text{c.}\,(m-r+1).$$

This is sometimes termed the number of *Variations without Repetitions* of the $r^{\text{th}}$ class.

292. Ex. Let $r$ be taken equal to the numbers 1, 2, 3, &c. in order; then we shall have

$$V_1 = m,$$
$$V_2 = m(m-1),$$
$$V_3 = m(m-1)(m-2), \text{ \&c.}$$

293. Cor. 1. Hence of a given number of things, the greater the number taken, the greater will be the number of variations formed.

Ex. Let there be six things given; then by the formulæ above investigated, we shall have

$$V_1 = 6,\ V_2 = 6.5 = 30,\ V_3 = 6.5.4 = 120,$$
$$V_4 = 6.5.4.3 = 360,\ V_5 = 6.5.4.3.2 = 720.$$

294. Cor. 2. If $r$ be supposed $= m$, or all the quantities be taken each time, we shall have

$$\text{the number of } \textit{Permutations} = m(m-1)(m-2) \,.\, \&\text{c.}\, 3 \,.\, 2 \,.\, 1$$
$$= 1 \,.\, 2 \,.\, 3 \,.\, \&\text{c.}\, (m-2)(m-1)\, m.$$

Ex. Required the number of changes which may be rung upon seven bells, taken all together.

Here, it is obvious that the number required is the same as the number of different permutations formed out of seven things, and therefore

$$= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040.$$

295. In article (290) it has been seen that the number of quantities within the brackets, in each case, does not comprise the one which is placed contiguously without it: in other words, that one thing has in each case been *subtracted* from the whole. If then instead of the operation of *subtraction*, that of *addition* be employed, we shall have corresponding thereto the following expressions:

$$a(a + b + c + \&\text{c.} + l + a),$$
$$b(a + b + c + \&\text{c.} + l + b),$$
$$c(a + b + c + \&\text{c.} + l + c),$$
$$\&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots$$
$$l(a + b + c + \&\text{c.} + l + l);$$

wherein, if the operations indicated be effected, there will obviously be $m(m+1)$ products:

again, on the same principle, if to these we annex each of the letters in succession, we get

$$aa(a + b + c + \&\text{c.} + l + a + a),$$
$$ab(a + b + c + \&\text{c.} + l + b + a),$$
$$ac(a + b + c + \&\text{c.} + l + c + a),$$
$$\&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots$$
$$al(a + b + c + \&\text{c.} + l + l + a);$$

and so on:

and it is manifest that the number of quantities thus formed will be $m(m+1)(m+2)$: wherefore similar steps will lead to the number of such quantities formed when the operation has been repeated any number of times.

296. The aggregate of the quantities constructed as in the last article, is styled the number of *Variations with Repetitions*, of the first, second, third, &c. orders; and the general formula may be investigated as follows.

Let $W_r$ denote the number of such variations, each comprising $r$ simple quantities; then it is obvious, from the nature of the case and from what has been said above, that

$$
\begin{aligned}
& W_r &&= (m+r-1)\,W_{r-1};\\
\text{similarly,}\quad & W_{r-1} &&= (m+r-2)\,W_{r-2},\\
& W_{r-2} &&= (m+r-3)\,W_{r-3},\\
& \&\text{c.} \ldots\ldots\ldots\ldots\ldots\ldots && \\
& W_2 &&= (m+1)\,W_1;
\end{aligned}
$$

and it is readily seen that $W_1 = m$: whence we obtain

$$
\begin{aligned}
W_r &= (m+r-1)(m+r-2)(m+r-3).\&\text{c.}\,(m+2)(m+1)m\\
&= m(m+1)(m+2).\&\text{c.}\,(m+r-2)(m+r-1),
\end{aligned}
$$

which is therefore the number of variations with repetitions of the $r^{\text{th}}$ class.

Ex. If there be five quantities, the number of variations with repetitions of the second class $= 5.6 = 30$: the number of the third class $= 5.6.7 = 210$: and these results may easily be verified.

## II. Combinations.

297. Def. The *Combinations* of any number of quantities or things are the different collections that can be formed out of them, by taking a certain number at a time, without regard to the order in which they are arranged.

Thus, of $a$, $b$, $c$, there will be three quantities $a$, $b$, $c$, formed by taking one at a time; three combinations $ab$, $ac$, $bc$, formed by taking two at a time; and one combination $abc$, made by taking all the three together.

298. *To find the number of Combinations that can be formed out of* m *things by always taking* r *of them together.*

Let $C_r$ be the number of combinations that can be formed out of $m$ things taken $r$ together; $V_r$ the corresponding number of variations: then since every combination of $r$ things taken all together, admits of $1.2.3.\&c.\ r$ permutations by (294), we shall have $1.2.3.\&c.\ r$ times the number of combinations equal to the corresponding number of variations: that is

$$(1.2.3.\&c.\ r)\ C_r = m(m-1)(m-2).\&c.\ (m-r+1), \text{ by (291)};$$

$$\text{whence } C_r = \frac{m(m-1)(m-2).\&c.\ (m-r+1)}{1.2.3.\&c.\ r}.$$

This is sometimes called the number of *Combinations without Repetitions* of the $r^{\text{th}}$ class.

299. Ex. If we make $r$ equal to the natural numbers $1.2.3.\&c.\ m$ in order, we shall obtain

$$C_1 = m;$$

$$C_2 = \frac{m(m-1)}{1.2};$$

$$C_3 = \frac{m(m-1)(m-2)}{1.2.3}; \ \&c.$$

$$C_m = \frac{m(m-1)(m-2).\&c.\ 3.2.1}{1.2.3.\&c.\ m} = 1.$$

300. Cor. 1. From the last article it appears that the numbers of combinations of $m$ things formed by taking one,

two, three, &c., $r$ together respectively, are the coefficients of the second, third, fourth, &c., $(r+1)^{\text{th}}$ terms of the expansion of the binomial $(1+x)^m$: and consequently the sum of all the combinations that can be formed, by taking 1, 2, 3, &c., $m$ of the quantities together, will be equal to the sum of all the coefficients of the expansion of $(1+x)^m$ diminished by the first; that is, to $2^m-1$, as appears from (195).

Also, since the coefficient of the first term of the expansion of $(1+x)^m$ is here wanting, it is manifest from (195) that the total number of odd combinations which can be formed out of $m$ things is greater by unity than the total number of even.

Ex. 1. Out of seven things proposed, we shall therefore have

$$C_1=7,\ C_2=21,\ C_3=35,\ C_4=35,\ C_5=21,\ C_6=7 \text{ and } C_7=1.$$

Ex. 2. Required the number of all the combinations that can be formed out of five things represented by $a$, $b$, $c$, $d$, $e$.

Here we have $m=5$; and therefore the number required

$$=2^5-1=32-1=31:$$

and if the quantities taken *singly* be excluded, the number

$$=31-5=26.$$

301. Cor. 2. Hence, also, the number of combinations of $m$ things formed by taking $r$ at a time will be the greatest when the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^m$ is the greatest: that is, by (197), when $r+1$ is the whole number equal to, or next greater than $\frac{1}{2}(m+1)$, and therefore when $r$ is that which is equal to, or next greater than $\frac{1}{2}(m-1)$.

Ex. Of five things, how many must be taken at a time, that the number of combinations may be the greatest possible?

If $r$ be the number taken, $r$ must be the whole number which is equal to, or next greater than $\frac{m-1}{2}$ or $\frac{5-1}{2}$ or 2: therefore, when either two or three are taken together, the number of combinations amounts to 10, and is greater than when any other number is taken.

302. Cor. 3. If in the formula deduced in (298) we substitute $m-r$ in the place of $r$, we shall, retaining the same kind of notation, have

$$C_{m-r} = \frac{m(m-1)(m-2).\&c.(r+1)}{1.2.3.\&c.(m-r)}$$

$$= \frac{m(m-1).\&c.(m-r+1)(m-r).\&c.(r+1)}{1.2.3.\&c.r.(r+1).\&c.(m-r)}$$

$$= \frac{m(m-1)(m-2).\&c.(m-r+1)}{1.2.3.\&c.r} = C_r,$$

by rejecting from the numerator and denominator such factors as are common to both.

The combinations belonging to the respective sets denoted by $C_r$ and $C_{m-r}$, are said to be *Supplementary* to each other.

Ex. In the first example of (300) where $m=7$, we have seen that

$$C_1 = C_6,\ C_2 = C_5 \text{ and } C_3 = C_4.$$

303. Cor. 4. If $C_r$ denote the number of combinations with repetitions, then, from (296), we have

$$(1.2.3.\&c.r)\,C_r = m(m+1)(m+2).\&c.(m+r-1):$$

$$\text{wherefore } C_r = \frac{m(m+1)(m+2).\&c.(m+r-1)}{1.2.3.\&c.r}.$$

304. *To find the number of different Permutations that can be formed out of* m *things taken all together, when* p *of them are of one sort,* q *of another, &c.*

Retaining the notation of (294), let us take $P$ to be the number of permutations required of the $m$ things, whereof $p$ are $a$'s, $q$ are $b$'s, &c.: then it is obvious, that if we supposed each of the $a$'s, $b$'s, &c. to be changed into a different letter, the whole number of permutations would be

$$P\,(1\,.\,2\,.\,3\,.\,\&c.\ p)\,(1\,.\,2\,.\,3\,.\,\&c.\ q)\,.\,\&c.:$$

but the whole number of permutations $=1.2.3.\&c.\ m$, by (294),

$$\text{whence } P=\frac{1\,.\,2\,.\,3\,.\,\&c.\ m}{(1\,.\,2\,.\,3\,.\,\&c.\ p)\,(1\,.\,2\,.\,3\,.\&c.\ q).\&c.}.$$

Ex. 1. Required the number of different permutations that can be formed out of the letters $aabbc$, taken all together.

Here are two $a$'s, two $b$'s and one $c$, so that the required number

$$P=\frac{1\,.\,2\,.\,3\,.\,4\,.\,5}{(1\,.\,2)\,(1\,.\,2)}=30\,;$$

which may easily be written down at length, and the result verified.

Ex. 2. Required the number of different permutations that can be formed out of the letters of the word *indifference*.

Here the whole number of letters $m=12$; and there are two $i$'s or $p=2$, two $n$'s or $q=2$, two $f$'s or $r=2$, and three $e$'s or $s=3$:

$$\text{whence we have } P=\frac{1.2.3.4.5.6.7.8.9.10.11.12}{(1.2)\,(1.2)\,(1.2)\,(1.2.3)}=9979200.$$

Ex. 3. Find the number of different permutations that can be formed out of $a^{m-r}b^r$ when written at length.

Here there are $m$ quantities, and $a$ and $b$ being repeated $m-r$ and $r$ times respectively, we have the required number

$$P = \frac{1 \cdot 2 \cdot 3 \cdot \&c.\ m}{\{1 \cdot 2 \cdot 3 \cdot \&c.\ (m-r)\}\ \{1 \cdot 2 \cdot 3 \cdot \&c.\ r\}}$$

$$= \frac{(m-r+1)\ (m-r+2) \cdot \&c.\ m}{1 \cdot 2 \cdot 3 \cdot \&c.\ r}$$

$$= \frac{m(m-1) \cdot \&c. (m-r+2)(m-r+1)}{1 \cdot 2 \cdot 3 \cdot \&c.\ r},$$

by rejecting from the numerator and denominator the factors common to both.

305. Cor. In the formula investigated, in (304) where

$$P = \frac{1 \cdot 2 \cdot 3 \cdot \&c.\ m}{(1 \cdot 2 \cdot 3 \cdot \&c.\ p)\ (1 \cdot 2 \cdot 3 \cdot \&c. q) \cdot \&c.},$$

it is manifest, that if $p$ represent the number of $b$'s, $q$ the number of $a$'s, &c. the value of the expression on the latter side still remains the same.

The quantities thus found are sometimes denominated *Complementary* to those which would be formed on the other hypothesis, their number being obviously the same.

306. *To find the numbers of Combinations and Variations that can be formed out of a certain number of things taken a given number together, when some are of one sort, some of another, &c.*

To avoid complication in the expressions, let us take a particular number of things, as $aaaabbbccd$, which may be written $a^4b^3c^2d$, and suppose five of them to be taken at a time: then the forms of the different combinations which can be made by always taking five of them together are

$$a^4\beta,\ a^3\beta^2,\ a^3\beta\gamma,\ a^2\beta^2\gamma:$$

now, of the first form are $a^4b$, $a^4c$, $a^4d$;

of the second are $a^3b^2$, $a^3c^2$, $b^3a^2$, $b^3c^2$;

of the third are $a^3bc$, $a^3bd$, $a^3cd$, $b^3ac$, $b^3ad$, $b^3cd$;

of the fourth are $a^2b^2c$, $a^2b^2d$, $a^2c^2b$, $a^2c^2d$, $b^2c^2d$:

therefore the number of combinations required is evidently 18: and similarly of other cases.

Hence, if the number of Variations that can be made out of each of the forms $a^4\beta$, $a^3\beta^2$, $a^3\beta\gamma$ and $a^2\beta^2\gamma$ be found, and multiplied by the number of combinations belonging to that form, the sum of the results will manifestly be the total number of Variations.

307. *To find the number of Combinations that can be formed out of* r *sets of things consisting of different numbers, by taking one out of each set for every such Combination.*

Let $a_1$, $b_1$, $c_1$, $d_1$, &c., $a_2$, $b_2$, $c_2$, $d_2$, &c., $a_3$, $b_3$, $c_3$, $d_3$, &c., $a_r$, $b_r$, $c_r$, $d_r$, &c., represent the different sets of things consisting of the different numbers $m_1$, $m_2$, $m_3$, &c. $m_r$: then, since every thing in the first set may be combined with every one in the second, the whole number of combinations of *two* things will manifestly be $m_1 m_2$.

Again, since with each of these combinations, every thing in the third set may be separately combined, we shall obviously have the number of combinations of *three* things, taken one out of each set, $= m_1 m_2 m_3$: and so on.

Whence by a continuation of this process, we shall manifestly find the whole number of combinations that can be formed out of the $r$ sets, by always taking one out of each set, to be $m_1\ m_2$ &c. $m_r$; which will therefore be so many combinations of things always taken $r$ together.

This may be illustrated as follows.

Connecting the things in the first, second, third, &c. sets together respectively, by means of the sign $+$, as under:

$$a_1 + b_1 + c_1 + \&c., \quad a_2 + b_2 + c_2 + \&c., \quad a_3 + b_3 + c_3 + \&c.; \quad \&c.$$

by multiplication of the first of these by the second, we obtain

$$a_2 a_1 + a_2 b_1 + a_2 c_1 + \&c. + b_2 a_1 + b_2 b_1 + b_2 c_1 + \&c.$$
$$+ c_2 a_1 + c_2 b_1 + c_2 c_1 + \&c.$$

which are the combinations by two and two, formed by always taking one out of each of the two sets.

Again, multiplying this result by $a_3 + b_3 + c_3 + \&c.$, we find the product to be

$$a_3 a_2 a_1 + a_3 a_2 b_1 + a_3 a_2 c_1 + \&c. + a_3 b_2 a_1 + a_3 b_2 b_1 + a_3 b_2 c_1 + \&c.$$
$$+ a_3 c_2 a_1 + a_3 c_2 b_1 + a_3 c_2 c_1 + \&c. + b_3 a_2 a_1 + \&c.$$

which obviously contains all the combinations similarly formed, by always selecting one out of each of the three sets: and so on.

Ex. There are four sets of things consisting of three, four, five and six individuals respectively; what number of different collections can there be made by always taking one out of each set?

In this instance $m_1 = 3$, $m_2 = 4$, $m_3 = 5$ and $m_4 = 6$;

whence the required number $= 3 . 4 . 5 . 6 = 360$.

308. Cor. If there be the same number of things in each set, we have

$$m_1 = m_2 = m_3 = \&c. = m_r;$$

and therefore the required number of combinations $= m^r{}_1$.

The quantity $m^r$ will manifestly be the number of terms in the result arising from the continued multiplication of $r$ polynomials, each consisting of $m$ terms.

Ex. How many combinations of three numbers can there be obtained by throwing three dice?

Each die having six faces, the question is to determine how many combinations of three things can be formed out of three sets, each consisting of six individual things, by always taking one out of each set:

$$\therefore\ m_1 = 6,\ m_2 = 6 \text{ and } m_3 = 6;$$

$$\text{whence the number required} = 6 \times 6 \times 6 = 216.$$

Of the subjects discussed in this chapter, the applications, hitherto given, have been what may perhaps be termed curious rather than useful: the following instances will, however, evince the practical utility of some of them.

309. Let it be required to determine the continued product of the $m$ simple factors $x + a$, $x + b$, $x + c$, &c., $x + l$.

Here we have $(x + a)(x + b) = x^2 + (a + b)x + ab$:

$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc:$$

$$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3$$
$$+ (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd:$$

and so on: and by an inductive process it may readily be shewn, that if the same kind of form be assumed to represent the continued product of $m - 1$ factors, it will likewise hold good for that of $m$ factors.

Wherefore, in the product required, it is obvious that the first term will be $x^m$: the second term will be $x^{m-1}$, having for its coefficient the sum of the combinations of the $m$ quantities $a$, $b$, $c$ &c., $l$, taken *one* at a time, and whose number is therefore $m$: the third term will be $x^{m-2}$, having its coefficient equal to the sum of the combinations of the $m$ quantities $a$, $b$, $c$ &c., $l$, taken *two* together, the number of which

will therefore be $\frac{m(m-1)}{1 \cdot 2}$: and so on: and the $n^{\text{th}}$ term will be $x^{m-n+1}$, with a coefficient equal to the sum of the combinations of the said $m$ quantities, formed by taking $n-1$ at a time, the number of which will therefore be

$$\frac{m(m-1)(m-2) \cdot \&c. (m-n+2)}{1 \cdot 2 \cdot 3 \cdot \&c. (n-1)};$$

and the last term will be the continued product of the $m$ quantities $a$, $b$, $c$, &c. $l$, and equal to $abc \cdot \&c.\ l$.

310. Cor. Let the $m$ quantities $a$, $b$, $c$, &c. $l$ be all equal to one another and to $a$, and then we shall manifestly have

$$(x+a)^m = x^m + m a x^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2} + \&c.,$$

which is the *Binomial Theorem* in the case where the index is a positive whole number.

311. Were it required to ascertain the nature and form of the continued product

$$(x_1 + a_1)(x_2 + a_2)(x_3 + a_3) \cdot \&c. \text{ to } m \text{ factors,}$$

we should readily observe, that every term of the result must necessarily consist of $m$ factors, and therefore all the terms must be homogeneous.

Also, when $m-1$ of the $x$'s are found in any term, *one* of the $a$'s must be involved with them: when $m-2$ of the $x$'s are involved, two of the $a$'s will also be found there, and so on: in other words, the $x$'s and $a$'s may be considered *complementary* to each other, the number of both together being in every term equal to $m$.

Now, from the formula investigated in article (304),

$$P = \frac{1 \cdot 2 \cdot 3 \cdot \&c.\ m}{(1 \cdot 2 \cdot 3 \cdot \&c.\ p)(1 \cdot 2 \cdot 3 \cdot \&c.\ q) \cdot \&c.},$$

the number of terms which are made up of $(m-1)$ $x$'s and *one a*, will be

$$\frac{1.2.3.\&c.\ m}{\{1.2.3.\&c.\ (m-1)\}\ \{1\}}=m:$$

the number of terms involving $(m-2)$ $x$'s and *two a*'s, will be

$$\frac{1.2.3.\&c.\ m}{\{1.2.3.\&c.\ (m-2)\}\ \{1.2\}}=\frac{m(m-1)}{1.2}:$$

the number of terms involving $(m-3)$ $x$'s and *three a*'s, will be

$$\frac{1.2.3.\&c.\ m}{\{1.2.3.\&c.\ (m-3)\}\ \{1.2.3\}}=\frac{m(m-1)(m-2)}{1.2.3}:$$

and so on:

and thus the total number of terms will be expressed by

$$1+m+\frac{m(m-1)}{1.2}+\frac{m(m-1)(m-2)}{1.2.3}+\&c.;$$

which, as appears from (195), is also equivalent to $2^m$.

312. Cor. 1. If it be required to find the nature and number of the terms of the continued product of $m$ multinomial factors, we shall observe that the terms, as before, will all be homogeneous; and the sum of the indices in every term being $m$, the magnitudes and number of such terms will be determined in the same manner from the expression

$$P=\frac{1.2.3.\&c.\ m}{(1.2.3.\&c.\ p)(1.2.3.\&c.\ q)(1.2.3.\&c.\ r).\&c.},$$

the values of $p$, $q$, $r$, &c. being always such that

$$p+q+r+\&c.=m.$$

313. Cor. 2. Hence also is readily deduced the Polynomial or Multinomial Theorem, which has been so fully considered in (211) and some of the subsequent articles.

314. The proposition explained in (295), and the formula investigated in (303), will readily lead to the determination of the number of the *Homogeneous Products* of different dimensions which can be formed out of the $m$ different quantities $a$, $b$, $c$, &c. $l$.

For, since we have seen in (303), that

$$C_r = \frac{m(m+1)(m+2).\&c.(m+r-1)}{1.2.3.\&c.\ r}:$$

if $r$ be supposed to be equal to 1, 2, 3, 4, &c. in succession, we shall find

$$C_1 = m, \quad C_2 = \frac{m(m+1)}{1.2},$$

$$C_3 = \frac{m(m+1)(m+2)}{1.2.3}, \quad C_4 = \frac{m(m+1)(m+2)(m+3)}{1.2.3.4}, \ \&c.$$

315. Cor. From the last article, by making $m$ equal to 2, 3, 4, &c. in succession, we learn that the number of terms in the developement of $(a+b)^r$

$$= \frac{2.3.4.\&c.(r+1)}{1.2.3.\&c.\ r} = r+1:$$

the number in the developement of $(a+b+c)^r$

$$= \frac{3.4.5.\&c.(r+2)}{1.2.3.\&c.\ r} = \frac{(r+1)(r+2)}{1.2};$$

the number in the developement of $(a+b+c+d)^r$

$$= \frac{4.5.6.\&c.(r+3)}{1.2.3.\&c.\ r} = \frac{(r+1)(r+2)(r+3)}{1.2.3};$$

and so on.

# CHAP. XI.

## *On the different Scales of Notation and the mode of performing Arithmetical operations in them, &c.*

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316. Def. Notation is the method of representing or expressing abstract numerical magnitudes; and it is divided into different *Scales* dependent upon the numbers and magnitudes of the figures employed.

### I. Integers.

317. *If* r *be any whole number, and* $a_0$, $a_1$, $a_2$, &c., $a_m$ *be integral quantities less than* r, *every number whatever may be represented in the following form*:

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \text{\&c.} + a_2 r^2 + a_1 r + a_0,$$

or

$$N = a_0 + a_1 r + a_2 r^2 + \text{\&c.} + a_{m-2} r^{m-2} + a_{m-1} r^{m-1} + a_m r^m.$$

For, let $N$ be divided by the greatest power of $r$ contained in it as $r^m$, and let the quotient be $a_m$ and the remainder $N_1$; then we shall have

$$N = a_m r^m + N_1:$$

again, let $N_1$ be divided by the greatest power of $r$ contained in it as $r^{m-1}$, and let the quotient and remainder be respectively $a_{m-1}$ and $N_2$; therefore, as before, we have

$$N_1 = a_{m-1} r^{m-1} + N_2;$$

$$\text{whence } N = a_m r^m + N_1$$
$$= a_m r^m + a_{m-1} r^{m-1} + N_2;$$

and so on:

and thus continuing the operation till the remainder becomes less than $r$, we shall evidently arrive at last at the form

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \&c. + a_2 r^2 + a_1 r + a_0,$$

which, by reversing the order of the terms, may also be written

$$N = a_0 + a_1 r + a_2 r^2 + \&c. + a_{m-2} r^{m-2} + a_{m-1} r^{m-1} + a_m r^m:$$

and because in each succeeding operation of division the greatest possible power of $r$ is taken, it follows that each of the quantities $a_0$, $a_1$, $a_2$, &c, $a_{m-2}$, $a_{m-1}$, $a_m$ is less than $r$.

The quantity $r$ is called the *Radix* or *Base* of the scale of notation, and the quantities $a_0$, $a_1$, $a_2$, &c, $a_{m-2}$, $a_{m-1}$, $a_m$ are termed the *Digits* belonging to that scale.

Hence any number consisting of $p$ figures or digits may be represented in the form

$$N = a_{p-1} r^{p-1} + a_{p-2} r^{p-2} + \&c. + a_2 r^2 + a_1 r + a_0:$$

and the greatest and least numbers, consisting of $p$ digits, will therefore be

$$(r-1) r^{p-1} + (r-1) r^{p-2} + \&c. + (r-1) r + (r-1)$$

and $r^{p-1}$ respectively, or $r^p - 1$ and $r^{p-1}$.

318. Cor. 1. If the order of the digits composing the number $N$ be inverted, and the corresponding number be $N'$, then we shall have the following formulæ:

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \&c. + a_2 r^2 + a_1 r + a_0,$$

$$N' = a_0 r^m + a_1 r^{m-1} + a_2 r^{m-2} + \&c. + a_{m-2} r^2 + a_{m-1} r + a_m;$$

whence, subtracting the latter from the former, we obtain

$$N - N' = a_m (r^m - 1) + a_{m-1} r (r^{m-2} - 1) + a_{m-2} r^2 (r^{m-4} - 1) + \&c.$$

which is universally divisible by $r-1$; and will be divisible by both $r-1$ and $r+1$ whenever $m$ is an even number.

319. Cor. 2. Since, in dividing by $r$, the remainder may be any quantity less than $r$, it follows that the digits in a system whose base is $r$ are 0, 1, 2, 3, &c., $r-1$; and consequently the number of digits cannot be either greater or less than the radix of the system.

320. Cor. 3. By assigning different values to $r$, the number $N$ will be expressed in different scales: thus,

| | | |
|---|---|---|
| if $r=2$, | the scale is termed the | *Binary;* |
| $r=3$, | ........................ | *Ternary;* |
| $r=4$, | ........................ | *Quaternary;* |
| $r=5$, | ........................ | *Quinary;* |
| $r=6$, | ........................ | *Senary;* |
| &c. | ........................ | &c.......... |
| $r=10$, | ........................ | *Denary;* |
| $r=11$, | ........................ | *Undenary;* |
| $r=12$, | ........................ | *Duodenary;* |
| &c. | ........................ | &c.......... |

321. Cor. 4. Whence we shall have the following digits or figures to be used in the respective scales, including the auxiliary digit 0 denominated zero:

In the binary, 0, 1;

In the ternary, 0, 1, 2;

In the quaternary, 0, 1, 2, 3;

In the quinary, 0, 1, 2, 3, 4;

In the senary, 0, 1, 2, 3, 4, 5;

&c........................

In the denary, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;

In the undenary, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, $t$;

In the duodenary, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, $t$, $u$;

&c..........................................

in the last two of which the letters $t$ and $u$ are supposed to represent the numbers *ten* and *eleven*, because we have no simple numerical symbol to answer that purpose.

Ex. In the binary scale, $101 = 1 \cdot 2^2 + 0 \cdot 2 + 1$.

In the senary scale, $453 = 4 \cdot 6^2 + 5 \cdot 6 + 3$.

In the denary scale, $2079 = 2 \cdot 10^3 + 0 \cdot 10^2 + 7 \cdot 10 + 9$.

In the duodenary scale, $3807 = 3 \cdot 12^3 + 8 \cdot 12^2 + 0 \cdot 12 + 7$.

From these examples it appears that in the expression of a number, every digit, in addition to its original value, possesses also a *local* value which depends upon the radix of the scale to which it belongs, and the digits are sometimes styled so many units of the first, second, third, &c. orders, dependent upon the situations which they respectively occupy.

Thus, in the second instance, the digit 4 represents $4 \cdot 6^2$; the digit 5 denotes $5 \cdot 6$ and the digit 3 retains its real value: so also 2079 in the denary scale denotes 2000, 70 and 9, the values of the figures 2 and 7 depending entirely upon their locality.

We shall suppose all numbers proposed to belong to the denary scale, unless the contrary be expressed.

322. Cor. 5. From the nature of the scales, as above explained, it is evident that all numbers represented in them may be transformed to the denary or common scale, by merely performing the operations indicated.

Ex. 1. The number 234 in the quinary scale is equivalent to

$$2 \cdot 5^2 + 3 \cdot 5 + 4$$

$$= 50 + 15 + 4 = 69,$$

in the denary, or common scale: this may be written

$$(234)_5 = (69)_{10}:$$

also, 6254 in the septenary scale is equivalent to

$$6 \cdot 7^3 + 2 \cdot 7^2 + 5 \cdot 7 + 4$$

$$= 2058 + 98 + 35 + 4 = 2195,$$

in the denary scale: that is, $(6254)_7 = (2195)_{10}$:

again, $9\,t\,507$ in the undenary scale is equivalent to

$$9 \cdot 11^4 + 10 \cdot 11^3 + 5 \cdot 11^2 + 0 \cdot 11 + 7$$

$$= 131769 + 13310 + 605 + 0 + 7 = 145691,$$

in the denary scale: or $(9t507)_{11} = (145691)_{10}$; and so on.

Ex. 2. Required the radix of the scale in which 425 is equivalent to two hundred and fifteen.

If $r$ denote the quantity required, we shall have

$$4r^2 + 2r + 5 = 215,$$

$$\text{or } r^2 + \tfrac{1}{2}r = 52\tfrac{1}{2}:$$

whence, by the solution of the quadratic, we find the values of $r$ to be 7 and $-7\frac{1}{2}$, the latter of which is excluded by the nature of the question: wherefore 7 is the required radix, as may easily be proved to be correct.

323. *Given the radix of the scale, to find the digits representing any proposed number in that scale.*

Let

$$N = a_m r^m + a_{m-1} r^{m-1} + a_{m-2} r^{m-2} + \text{\&c.} + a_2 r^2 + a_1 r + a_0,$$

in which $a_m$, $a_{m-1}$, $a_{m-2}$, &c, $a_2$, $a_1$, $a_0$ are the digits required to be found, the magnitude of $r$ being supposed known:

then, if we divide both sides of this equation by $r$, the quotient will be $a_m r^{m-1} + a_{m-1} r^{m-2} + a_{m-2} r^{m-3} + \text{\&c.} + a_2 r + a_1$, with a remainder $a_0$, which is the first digit on the right hand:

again, repeating the same operation with this result, the quotient is obviously $a_m r^{m-2} + a_{m-1} r^{m-3} + a_{m-2} r^{m-4} + \&c. + a_2$, with a remainder $a_1$, the second digit from the right hand; and so on: from which it appears that the digits beginning at the right hand are the remainders after the successive divisions of the number by the radix of the scale proposed.

Ex. Express the common number 75432 in the senary and duodenary scales.

In the former case $r=6$ and in the latter $r=12$; wherefore we have the two following operations;

$$\text{(1).}\quad \begin{array}{r|l} 6\,)\ 75432 & \\ \hline 6\,)\ 12572 & 0=a_0, \\ \hline 6\,)\ 2095 & 2=a_1, \\ \hline 6\,)\ 349 & 1=a_2, \\ \hline 6\,)\ 58 & 1=a_3, \\ \hline 6\,)\ 9 & 4=a_4, \\ \hline 6\,)\ 1 & 3=a_5, \\ \hline 0 & 1=a_6; \end{array} \qquad \text{(2).}\quad \begin{array}{r|l} 12\,)\ 75432 & \\ \hline 12\,)\ 6286 & 0=a_0, \\ \hline 12\,)\ 523 & t=a_1, \\ \hline 12\,)\ 43 & 7=a_2, \\ \hline 12\,)\ 3 & 7=a_3, \\ \hline 0 & 3=a_4; \end{array}$$

whence the common number 75432 is represented in the senary scale by 1341120, and in the duodenary by $377t0$: that is,

$$(75432)_{10} = (1341120)_6 \text{ and } (75432)_{10} = (377t0)_{12}:$$

and it may be observed generally, that the greater the radix of the scale proposed, the less will be the number or magnitude of the digits required to express a given number.

These results are easily verified; for, by performing the operations implied in the respective scales, we have

$$1 . 6^6 + 3 . 6^5 + 4 . 6^4 + 1 . 6^3 + 1 . 6^2 + 2 . 6 + 0 = 75432;$$

$$\text{and } 3 . 12^4 + 7 . 12^3 + 7 . 12^2 + 10 . 12 + 0 = 75432.$$

324. Cor. Hence, to transform a number from a scale whose radix is $r$ to another whose radix is $r'$, we have merely to express it in the common or denary scale by (322), and thence to find its expression in the scale whose radix is $r'$ by the last article.

Ex. Convert 3256 from a scale whose radix or local value is 7, to one whose local value is 12.

First, 3256 in the septenary scale is equivalent to

$$3 . 7^3 + 2 . 7^2 + 5 . 7 + 6 = 1168 \text{ in the denary:}$$

then, by (323), we have

$$\begin{array}{rl}
12 \,)\, 1168 & \\
\hline
12 \,)\, 97 & 4 = a_0, \\
\hline
12 \,)\, 8 & 1 = a_1, \\
\hline
0 & 8 = a_2;
\end{array}$$

wherefore 3256 in the septenary scale, when expressed in the duodenary scale, becomes 814, which may easily be verified.

325. *Every number whatever is composed of some number of the terms of the geometrical series*, 1, 2, $2^2$, $2^3$, &c., *indefinitely continued.*

For, in the binary scale of notation, we shall manifestly have

$$N = a_m 2^m + a_{m-1} 2^{m-1} + a_{m-2} 2^{m-2} + \text{\&c.} + a_2 2^2 + a_1 2 + a_0,$$

where $N$ may be any number whatever, and $a_m$, $a_{m-1}$, $a_{m-2}$, &c., $a_2$, $a_1$, $a_0$ are each less than 2, and must therefore be either 0 or 1:

that is, none of the terms of the progression are taken more than once, and consequently all numbers may be composed out of the sums of them by assigning proper values to $m$.

Ex. Express 37 by means of the terms of the series $1, 2, 2^2, 2^3$, &c.

First, to transform 37 into the binary scale, we have

$$\begin{array}{l|ll}
2) & 37 & \\
\hline
2) & 18 & 1 = a_0, \\
\hline
2) & 9 & 0 = a_1, \\
\hline
2) & 4 & 1 = a_2, \\
\hline
2) & 2 & 0 = a_3, \\
\hline
2) & 1 & 0 = a_4, \\
\hline
2) & 0 & 1 = a_5;
\end{array}$$

therefore, 37 in the common scale is equivalent to 100101 in the binary scale which is expressed by $2^5 + 2^2 + 1$.

326. Cor. If the number of terms of the series be limited and represented by $n$, then the greatest number that can be expressed by it will manifestly be its sum, or

$$1 + 2 + 2^2 + \&c. \text{ to } n \text{ terms} = 2^n - 1.$$

327. *Every number whatever may be formed by the sums and differences of some of the terms of the geometrical progression* $1, 3, 3^2, 3^3$, &c. *indefinitely extended.*

For, by representing any number in the ternary scale of notation, we have

$$N = a_m 3^m + a_{m-1} 3^{m-1} + a_{m-2} 3^{m-2} + \&c. + a_2 3^2 + a_1 3 + a_0,$$

in which each of the coefficients $a_m, a_{m-1}, a_{m-2}$, &c., $a_2, a_1, a_0$ being less than 3, must manifestly be 2, 1 or 0.

If every one of the coefficients be either 1 or 0, the proposition is evident: but if one or more of them be 2, so that

$$N = 2 \cdot 3^m + 2 \cdot 3^{m-1} + 2 \cdot 3^{m-2} + \&c. + 2 \cdot 3^2 + 2 \cdot 3 + 2,$$

then
$$\begin{aligned}
2 \cdot 3^m &= (3-1)\, 3^m &&= 3^{m+1} - 3^m; \\
2 \cdot 3^{m-1} &= (3-1)\, 3^{m-1} &&= 3^m - 3^{m-1}; \\
2 \cdot 3^{m-2} &= (3-1)\, 3^{m-2} &&= 3^{m-1} - 3^{m-2}; \\
&\&c. \ldots\ldots\ldots\ldots\ldots\ldots \\
2 \cdot 3^2 &= (3-1)\, 3^2 &&= 3^3 - 3^2; \\
2 \cdot 3 &= (3-1)\, 3 &&= 3^2 - 3; \\
2 &= (3-1) &&= 3 - 1:
\end{aligned}$$

whence, by substitution, we obtain

$$N = 3^{m+1} - 1:$$

and a similar method of proceeding will answer in all other cases.

Ex. Let it be required to express 206 by means of the sums and differences of the terms of the geometrical progression 1, 3, $3^2$, $3^3$, &c.

First, to transform 206 from the denary to the ternary scale of notation, we have

$$\begin{array}{l|rl}
3) & 206 & \\ \hline
3) & 68 & 2 = a_0, \\ \hline
3) & 22 & 2 = a_1, \\ \hline
3) & 7 & 1 = a_2, \\ \hline
3) & 2 & 1 = a_3, \\ \hline
 & 0 & 2 = a_4;
\end{array}$$

therefore 206 expressed in the scale whose radix is 3, becomes

$$
\begin{aligned}
21122 &= 2 \,.\, 3^4 + 1 \,.\, 3^3 + 1 \,.\, 3^2 + 2 \,.\, 3 + 2 \\
&= 3^5 - 3^4 + 3^3 + 3^2 + 3^2 - 3 + 3 - 1 \\
&= 3^5 - 3^4 + 3^3 + 2 \,.\, 3^2 - 1 = 3^5 - 3^4 + 2 \,.\, 3^3 - 3^2 - 1 \\
&= 3^5 - 3^4 + 3^4 - 3^3 - 3^2 - 1 = 3^5 - (3^3 + 3^2 + 1),
\end{aligned}
$$

which, by (322), may easily be proved to be correct.

328. Cor. Hence the greatest number that can be expressed by $n$ terms of this series will obviously be their sum, which $= \frac{1}{2}(3^n - 1)$; though numbers less than this may require all the terms in the series.

Ex. Let us take the number 700, which is easily proved, as in the last article, to be equal to $(3^6 + 1) - (3^3 + 3)$, so that terms as far as the seventh have been employed, although 700 is much less than $\frac{1}{2}(3^7 - 1)$ or 1093.

329. Similar considerations will, in some instances, enable us to obtain analogous results in the other scales.

Thus, 51 in the denary scale will be represented by 303 in the quaternary, which

$$
\begin{aligned}
&= 3 \,.\, 4^2 + 0 \,.\, 4 + 3 \\
&= (4 - 1)\, 4^2 + (4 - 1) \\
&= 4^3 - 4^2 + 4 - 1 \\
&= (4^3 + 4) - (4^2 + 1).
\end{aligned}
$$

330. *To perform the Arithmetical operations of Addition, Subtraction, &c. in a Scale of Notation whose Radix is* r.

From what has been already said respecting the different scales of notation, and from the nature of the proposed operations, it is obvious that the processes will be similar to those

used in the common scale, with this difference only, that $r$ must here be used in the cases wherein the number 10 would be applied, did the numbers proposed belong to the common scale.

This will, however, be best illustrated by examples.

Ex. 1. Find the sum and difference of the numbers 45324502 and 25405534 in the senary scale.

First, arranging them as in the common scale, we have

$$\begin{array}{r} 45324502 \\ 25405534 \\ \hline \end{array}$$

$$\text{therefore the sum} = 115134440,$$

obtained by adding the numbers in vertical lines, as in common arithmetic, and carrying 1 for every 6 contained in the results and putting down the excesses above it:

$$\begin{array}{rr} \text{again,} & 45324502 \\ & 25405534 \\ \hline \end{array}$$

$$\text{therefore the difference} = 15514524,$$

which is found by subtraction, where we always borrow 6 when the digit in the lower line exceeds that in the upper, and add 1 to the next digit in the lower line for it.

Ex. 2. Required the product of 2483 and 589 in the undenary scale.

$$\begin{array}{rr} \text{Here, the multiplicand} = & 2483 \\ \text{the multiplier} = & 589 \\ \hline & 1t985 \\ & 18502\phantom{0} \\ & 11184\phantom{00} \\ \hline \end{array}$$

$$\text{therefore the product} = 13122t5,$$

and this is found by proceeding as in ordinary multiplication, by carrying 1 at every 11, the letter $t$ here denoting 10.

Ex. 3. To divide 1184323 by 589 in the duodenary scale, we have

$$
\begin{array}{l}
589)\ 1184323\ (2483 = \text{the quotient;} \\
\quad\ \ u56 \\
\quad\ \ \overline{22t3} \\
\quad\ \ 1tu0 \\
\quad\ \ \ \overline{3u32} \\
\quad\ \ \ 39t0 \\
\quad\ \ \ \ \overline{1523} \\
\quad\ \ \ \ 1523 \\
\quad\ \ \ \ \overline{\phantom{1523}}
\end{array}
$$

and here $t$ and $u$ represent 10 and 11, and the operation is conducted with reference to 12, as it is ordinarily done with respect to 10.

Ex. 4. Involution and Evolution are performed in a manner precisely similar; thus, in the senary scale,

$$(2405)^2 = (2405) \times (2405) = 11122441,$$

as in example (2).

Again, to extract the square root of 11122441 in the same scale, we have, after pointing as in common arithmetic,

$$
\begin{array}{rl}
 & 1\dot{1}1\dot{2}2\dot{4}4\dot{1}\ (2405 = \text{the square root.} \\
 & 4 \\
 & \overline{\phantom{4}} \\
44) & 312 \\
 & 304 \\
 & \overline{\phantom{304}} \\
5205) & 42441 \\
 & 42441 \\
 & \overline{\phantom{42441}}
\end{array}
$$

331. Cor. If $N = a_m r^m + a_{m-1} r^{m-1} + \&c. + a_1 r + a_0$, and both members of the equation be multiplied by $r^n$, we shall have

$$N r^n = a_m r^{m+n} + a_{m-1} r^{m+n-1} + \&c. + a_1 r^{n+1} + a_0 r^n,$$

so that each of the last $n$ digits is zero: in other words, a number may be multiplied by any power of the radix by affixing to it as many zeros as there are units in its index: and conversely.

332. *Given the numbers of digits in two numbers, to find the numbers of digits in their Sum and Difference.*

Let $p$ and $q$ be the numbers of digits composing $N$ and $N'$ respectively, whereof $p$ is greater than $q$; then, by (317), we shall have

$$N = a_0 + a_1 r + a_2 r^2 + \&c. + a_{q-2} r^{q-2} + a_{q-1} r^{q-1} + \&c. + a_{p-1} r^{p-1},$$

$$\text{and } N' = b_0 + b_1 r + b_2 r^2 + \&c. + b_{q-2} r^{q-2} + b_{q-1} r^{q-1}:$$

whence, by addition and subtraction, we obtain

$$N \pm N' = (a_0 \pm b_0) + (a_1 \pm b_1) r + (a_2 \pm b_2) r^2 + \&c.$$

$$+ (a_{q-2} \pm b_{q-2}) r^{q-2} + (a_{q-1} \pm b_{q-1}) r^{q-1} + \&c. + a_{p-1} r^{p-1}:$$

which will therefore generally comprise $p$ digits.

If $q = p - 1$, then since $a_{q-1} + b_{q-1}$ may be equal to, or greater than $r$, the expression $(a_{q-1} + b_{q-1}) r^{q-1} + a_{p-1} r^{p-1}$ may become equal to $c_q r^q + a_{p-1} r^{p-1}$ or $(c_{p-1} + a_{p-1}) r^{p-1}$, from which it is evident, that if $c_{p-1} + a_{p-1}$ be equal to, or greater than $r$, the last term will become of the form $d_p r^p$, and thus the sum $N + N'$ may contain $p + 1$ digits.

Similar reasoning may be applied, when the lower sign is used, to shew that the difference $N - N'$ may contain only $p - 1$ digits; and the same holds good in both cases when $q = p$.

Ex. 1. Let $N = 14263056$ and $N' = 5036425$, in the septenary scale; then $p = 8$ and $q = 7$:

also, $N + N' = 22332514$, which contains eight digits;

and $N - N' = 6223331$, which contains seven digits.

Ex. 2. Suppose $N = u23576$ and $N' = 93895$ in the duodenary scale; $\therefore p = 6$ and $q = 5$:

also, $N + N' = uu724u$, containing seven digits;

and $N - N' = t4u8t1$, containing six digits.

Similarly, if there be several numbers $N$, $N'$, $N''$, &c.

333. *Given the number of digits contained in each of two numbers, to find the number of digits constituting their Product.*

As in the last article, let $N$ and $N'$ consist of $p$ and $q$ digits respectively; then we shall have

$$N = a_{p-1}r^{p-1} + a_{p-2}r^{p-2} + \text{\&c.} + a_1 r + a_0, \text{ and}$$
$$N' = b_{q-1}r^{q-1} + b_{q-2}r^{q-2} + \text{\&c.} + b_1 r + b_0:$$

whence, multiplying these quantities together, we obtain

$$NN' = a_{p-1}b_{q-1}r^{p+q-2} + a_{p-2}b_{q-1}r^{p+q-3} + \text{\&c.}$$
$$+ a_{p-1}b_{q-2}r^{p+q-3} + \text{\&c.};$$

from which we infer, by (317), that the product $NN'$ must always consist of $p + q - 1$ digits at least.

Also, since each of the digits is necessarily less than $r$, the product of any two of them must always be less than $r^2$, but may be greater than $r$, and thence it follows that

$NN'$ must be less than $r^{p+q}$, but may be greater than $r^{p+q-1}$:

that is, the product $NN'$ consists of fewer digits than $p + q + 1$; or, in other words, cannot comprise more digits than $p + q$.

Ex. 1. Let $\left.\begin{array}{l} N = 83875 \\ N' = \quad 864 \end{array}\right\}$ in the nonary scale, where $p=5$ and $q=3$;

$$\begin{array}{r} 366832 \\ 555803\phantom{0} \\ 744764\phantom{00} \\ \hline \end{array}$$

$\therefore$ $NN' = 81523362$, which consists of eight digits.

Ex. 2. If the numbers be expressed in the undenary scale, and $N = 123t$, $N' = 23t4$; then, we have

$$\begin{array}{r} N = 123t \\ N' = 23t4 \\ \hline 4947 \\ 11161\phantom{0} \\ 3708\phantom{00} \\ 2479\phantom{000} \\ \hline \end{array}$$

$\therefore$ $NN' = 2955157$,

in which the number of digits $= 7 = 4 + 4 - 1$.

334. Cor. If the three numbers $N$, $N'$ and $N''$ consist of $p$, $q$ and $s$ digits respectively: then, since $NN'$ comprises $p + q - 1$ or $p + q$ digits, it is obvious that

$NN'N''$ will comprise either $(p + q - 1) + s - 1$, $(p + q - 1) + s$, or $p + q + s$ digits :

that is, $NN'N''$ may consist of $p + q + s - 2$, $p + q + s - 1$, or $p + q + s$ digits:

and the same kind of reasoning may be extended to the product of any number of quantities whatever.

335. *Given the number of digits comprised in each of two numbers, to find the number of digits in their Quotient.*

Let $N$ and $N'$ denote the two numbers, consisting of $p$ and $q$ digits respectively, whereof $N$ is the greater; and let $Q$ be the quotient arising from the division, so that

$$\frac{N}{N'} = Q \quad \text{or} \quad N = QN':$$

then, since $N$ consists of $p$ digits, it is obvious that $QN'$ must contain the same number: let $Q$ contain $x$ digits, then by (333) the number of digits in $QN'$ cannot be

$$> q + x \quad \text{nor} \quad < q + x - 1:$$

whence it follows, that $p$ cannot be

$$> q + x \quad \text{nor} \quad < q + x - 1:$$

from which inequalities we conclude that $x$, or the number of digits in the quotient $\frac{N}{N'}$, cannot be

$$< p - q \quad \text{nor} \quad > p - q + 1.$$

Ex. In the quinary scale, let $N = 20101$ and $N' = 213$; then we have

$$\begin{array}{r l}
213)\,20101\,(42 & = \text{the quotient;} \\
1412\phantom{1} & \\
\hline
431 & \\
431 & \\
\hline
\end{array}$$

and the number of digits in the quotient is $2 = 5 - 3$.

336. *Given the number of digits constituting any number, to find the number of digits in its Square, Cube, &c.*

Let the number $N$ consist of $p$ digits in any scale of notation; then it is manifest that the number of digits in $N^2$

or $N \times N$ cannot be

$$< p+p-1 \text{ or } 2p-1, \text{ nor } > p+p \text{ or } 2p,$$

as appears from (333).

Again, the number of digits in $N^3$ or $N \times N \times N$ may, by (334), be $p+p+p-2$, or $p+p+p-1$, or $p+p+p$: that is, the cube of $N$ may comprise

$$3p-2, \quad 3p-1 \text{ or } 3p \text{ digits:}$$

and a continuation of the same process will prove that the number of digits in $N^4$ may be

$$4p-3, \quad 4p-2, \quad 4p-1 \text{ or } 4p,\text{: \&c.;}$$

and generally that in $N^m$ it must be one of the quantities

$$mp-(m-1), \quad mp-(m-2), \quad mp-(m-3), \text{ \&c., } mp.$$

Ex. Supposing $N = 1354$ in the senary scale, we shall have

$$N^2 = (1354)^2 = 2425204,$$

which has $7 = 2.4-1$ digits:

$$N^3 = (1354)^3 = 4315231544,$$

which comprises $10 = 3.4-2$ digits; and so on.

337. *Given the number of digits forming any number, to find the number of digits in its Square Root, Cube Root, &c.*

Let the number $N$ consist of $p$ digits: and suppose its square root or $\sqrt{N}$ to comprise $x$ digits; then it is manifest from the last article, that $N$ cannot consist of

fewer than $2x-1$, nor of more than $2x$ digits:

that is, $p$ cannot be $< 2x-1$ nor $> 2x$:

and thence $x$ cannot be $> \dfrac{p+1}{2}$ nor $< \dfrac{p}{2}$.

Again, if we suppose $\sqrt[3]{N}$ to consist of $y$ digits, it will follow that $N$ must have $3y-2$, $3y-1$ or $3y$ digits: that is, we must have $p=3y-2$, $p=3y-1$ or $p=3y$, from which are readily deduced

$$y=\frac{p+2}{3},\quad y=\frac{p+1}{3}\quad\text{or}\quad y=\frac{p}{3};$$

and so on, for higher roots.

Ex. Let $N=132221$ in the octenary scale: then, by the ordinary process, we find

$$\sqrt{N}=\sqrt{132221}=327,$$

which comprises $3=\frac{1}{2}$ of $6$ digits.

338. Cor. From this proposition are deduced the rules for *pointing* in the extraction of the square, cube, &c. roots of numbers.

Thus, if a number consist of $p$ digits, it has been seen in (337) that its square root will comprise $\frac{1}{2}p$ digits if $p$ be even, and $\frac{1}{2}(p+1)$ digits if $p$ be odd: whence, if a point be placed over every alternate figure, beginning with the units' place, the number of points will obviously be the same as the number of digits in the root.

Again, from (337) it appears that the cube root will have $\frac{1}{3}p$, $\frac{1}{3}(p+1)$ or $\frac{1}{3}(p+2)$ digits, according as $p$ is a multiple of 3, or leaves the remainders 2 and 1 when divided by it: and hence in this case, the points placed over every third digit, beginning with that in the place of units, will indicate the number of digits in the corresponding root; and the same mode of proceeding will shew, that, in extracting the $m^{\text{th}}$ root, the number of points obtained by placing one over every $m^{\text{th}}$ figure, beginning with the units, will be that of the digits in the root.

339. *Given a number written in any scale of notation, to find the remainder arising from its division by any number either greater or less than the base of the system.*

Let $r$ be the radix of the scale of notation, $a_m$, $a_{m-1}$, &c., $a_1$, $a_0$ the digits; then we have

$$N = a_m r^m + a_{m-1} r^{m-1} + \&c. + a_2 r^2 + a_1 r + a_0;$$

also, since $r = r - d + d$, the number $N$ may be represented in the following form:

$$N = a_m \{(r-d)+d\}^m + a_{m-1}\{(r-d)+d\}^{m-1} + \&c.$$
$$+ a_2\{(r-d)+d\}^2 + a_1\{(r-d)+d\} + a_0,$$

from which, if the expansions be effected, we shall manifestly obtain

$$N = a_0 + a_1 d + a_2 d^2 + \&c. + a_{m-2} d^{m-2} + a_{m-1} d^{m-1} + a_m d^m + P,$$

where $P$ involves $r - d$ and its powers combined with the indices $m$, $m-1$, $m-2$, &c. and the powers of $d$ as factors:

wherefore, if $N$ be divided by $r-d$, it will obviously leave the same remainder as would be obtained by dividing

$$a_0 + a_1 d + a_2 d^2 + \&c. + a_{m-2} d^{m-2} + a_{m-1} d^{m-1} + a_m d^m \text{ by } r-d.$$

Consequently, if the remainder be 0 when $N$ is divided by $r-d$, there will likewise be no remainder when this last quantity is divided by it; or, in other words, the quantity

$$a_0 + a_1 d + a_2 d^2 + \&c. + a_{m-2} d^{m-2} + a_{m-1} d^{m-1} + a_m d^m$$

will be a multiple of $r-d$.

340. Cor. If the algebraical sign of $d$ be changed, we shall have

$$N = a_0 - a_1 d + a_2 d^2 - \&c. \pm a_{m-2} d^{m-2} \mp a_{m-1} d^{m-1} \pm a_m d^m + P,$$

where $P$ is made up of $r+d$ and its powers together with combinations of $m$, $m-1$, $m-2$, &c. and the powers of $d$, and the upper or lower sign is used according as $m$ is even or odd: whence, in this case, $N$ when divided by $r+d$, will leave the same remainder as the expression

$$a_0 - a_1 d + a_2 d^2 - \&\text{c.} \pm a_{m-2} d^{m-2} \mp a_{m-1} d^{m-1} \pm a_m d^m$$

leaves, when divided by the same quantity.

Ex. 1. If we suppose $d=1$, then from (339) we shall have

$$N = a_0 + a_1 + a_2 + \&\text{c.} + a_{m-2} + a_{m-1} + a_m + P;$$

from which it appears that when $N$ is divided by $r-1$, it leaves the same remainder as when the sum of its digits is divided by $r-1$; and consequently whenever $N$ is a multiple of $r-1$, the sum of the digits which compose it will also be a multiple of the same quantity.

In the denary or common scale of notation wherein $r=10$, we have $r-1=10-1=9$: and consequently any common number and the sum of its digits when divided by 9, leave the same remainder.

Hence, also, if from any number the sum of its digits be subtracted, the remainder will be divisible by 9.

Ex. 2. The instance last given furnishes us with a demonstration of the proofs of arithmetical multiplication, &c. by *Casting out the Nines.*

Let $A$ the multiplicand contain $p$ nines with a remainder $\alpha$, and $B$ the multiplier contain $q$ nines with a remainder $\beta$;

$$\text{then } A = 9p + \alpha,$$
$$\text{and } B = 9q + \beta;$$

$$\therefore AB = 81pq + 9q\alpha + 9p\beta + \alpha\beta$$
$$= 9\{9pq + q\alpha + p\beta\} + \alpha\beta:$$

whence it is obvious that $AB$, or the sum of the digits in $AB$, divided by 9 leaves the same remainder as $\alpha\beta$, or the sum of its digits divided by 9 leaves: hence the rule as exemplified below.

To multiply 27354 by 2687, we have

$$\begin{aligned} A &= 27354, \ \therefore\ \alpha = 3, \\ B &= \ \ 2687, \ \therefore\ \beta = 5; \end{aligned}$$

$$\begin{array}{r} 191478\phantom{000} \\ 218832\phantom{00} \\ 164124\phantom{0} \\ 54708 \\ \hline \end{array}$$

$$\therefore\ AB = 73500198, \text{ and } \alpha\beta = 15:$$

and it is easily seen that the remainders arising from the division of the sums of the digits in $AB$ and $\alpha\beta$, are both 6; from which it is inferred that the operation is correctly performed: and it can be erroneous only by some multiple of 9, or in the placing of its different parts.

By assigning different values to $r$, a similar method of proof is applicable in the other scales.

Ex. 3. If in the corollary to the proposition we make $d = 1$, then will

$$N = a_0 - a_1 + a_2 - \text{\&c.} \pm a_{m-2} \mp a_{m-1} \pm a_m + P,$$

where $P$ is divisible by $r + 1$; and from this it appears that the divisions of the two quantities

$$N \text{ and } a_0 - a_1 + a_2 - \text{\&c.} \pm a_{m-2} \mp a_{m-1} \pm a_m \text{ by } r + 1,$$

leave the same remainder.

Hence, also, if the difference between the sums of the digits in the odd and even places respectively of any number be divisible by $r + 1$, the number itself will be divisible by the same quantity.

Wherefore, likewise, if the sum of the digits in the odd places be subtracted from, and the sum of those in the even places be added to, any number, the result will be a multiple of $r+1$.

In the common scale of notation all these properties belong to $10+1$ or $11$, and from them might be derived a mode of proof similar to that explained in the last example.

Ex. 4. Let $d=2$; then by the proposition, we have

$$N = a_0 + a_1 2 + a_2 2^2 + \&c. + a_{m-2} 2^{m-2} + a_{m-1} 2^{m-1} + a_m 2^m + P,$$

wherein $P$ is a multiple of $r-2$:

wherefore if the two quantities

$N$ and $a_0 + a_1 2 + a_2 2^2 + \&c. + a_{m-2} 2^{m-2} + a_{m-1} 2^{m-1} + a_m 2^m$,

be both divided by $r-2$, the remainders will in all cases be equal.

In the common scale, where $r=10$, we shall evidently have $N$ divisible by 8 when $a_0 + 2a_1 + 4a_2$ is so divisible, since the succeeding terms are all multiples of $2^3$ or 8.

Ex. 5. If $d=-2$, the general formula of the corollary gives

$$N = a_0 - a_1 2 + a_2 2^2 - \&c. \pm a_{m-2} 2^{m-2} \mp a_{m-1} 2^{m-1} \pm a_m 2^m + P,$$

the last term being a multiple of $r+2$: and from this we conclude that in the common scale, if the digits beginning from the place of units, be successively multiplied by 1, 2, $2^2$, $2^3$, &c. and the difference of the odd and even terms be divisible by 12, the number itself is divisible by 12.

Ex. 6. Let $d=3$; then, by means of (339), we get

$$N = a_0 + a_1 3 + a_2 3^2 + \&c. + a_{m-2} 3^{m-2} + a_{m-1} 3^{m-1} + a_m 3^m + P,$$

and in this case $P$ is a multiple of $r-3$:

wherefore in the denary scale of notation the number $N$ will be divisible by 7 when

$$a_0 + a_1 3 + a_2 3^2 + \&c. + a_{m-2} 3^{m-2} + a_{m-1} 3^{m-1} + a_m 3^m$$

is divisible by 7, and the contrary.

Ex. 7. If we make $d=3$, the corollary gives

$$N = a_0 - a_1 3 + a_2 3^2 - \&c. \pm a_{m-2} 3^{m-2} \mp a_{m-1} 3^{m-1} \pm a_m 3^m + P;$$

from which we draw the same conclusions respecting the series 1, 3, $3^2$, $3^3$, &c. and the number 13, as were deduced respecting the series 1, 2, $2^2$, $2^3$, &c. and the number 12, in the last example.

By assigning to $d$ as used in articles (339) and (340), the values 4, 5, 6, &c. in succession, a variety of results common to all the scales may readily be deduced, and then applied to the numbers in common use by making $r=10$.

## II. Fractions.

341. Def. The fundamental principles of Numeration already explained, shewing that the local value of every digit increases $r$ fold as we advance towards the left hand, if the radix of the scale be $r$, it will follow that if the digits be taken in the contrary order, their local values must decrease in the same proportion. Hence therefore the local value of each of the digits in succession to the right of the units place becomes $r$ times less than that of the one which immediately precedes it.

If therefore we have the general algebraical formula,

$$N = a_m r^m + \&c. + a_2 r^2 + a_1 r + a_0 r^0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \&c.,$$

it is obvious that the quantities to the left of $a_0 r^0$ comprising units of orders superior to the first, will be whole numbers, whilst those, on the right of the same term, being of local values

inferior to the first, designate so many fractions: and in a quantity consisting of both it is usual to separate the integral part from that which is fractional by means of a point.

Thus, in the ternary scale, we shall have

$$120.21 = 1.3^2 + 2.3^1 + 0.3^0 + 2.3^{-1} + 1.3^{-2}$$

$$= 1.3^2 + 2.3^1 + 0.3^0 + \frac{2}{3^1} + \frac{1}{3^2}:$$

again, in the septenary scale, we have, in the same manner,

$$52.3406 = 5.7^1 + 2.7^0 + 3.7^{-1} + 4.7^{-2} + 0.7^{-3} + 6.7^{-4}$$

$$= 5.7^1 + 2.7^0 + \frac{3}{7^1} + \frac{4}{7^2} + \frac{0}{7^3} + \frac{6}{7^4}:$$

so likewise, in the denary or common scale,

$$2.71828 = 2.10^0 + 7.10^{-1} + 1.10^{-2} + 8.10^{-3} + 2.10^{-4} + 8.10^{-5}$$

$$= 2.10^0 + \frac{7}{10^1} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{2}{10^4} + \frac{8}{10^5};$$

in which each of the digits has manifestly an absolute value as well as one dependent upon the situation in which it is placed.

342. Cor. Hence, in order to multiply or divide a quantity by any power of the radix, we have only to remove the separating point to the right or left as many places as there are units contained in its index, since by such a step the denomination of every digit is increased or diminished in the proposed ratio.

343. *To express a fraction of the kind just explained, by means of a Vulgar Fraction.*

Let $N = a_{-1} r^{-1} + a_{-2} r^{-2} + a_{-3} r^{-3} + \&c. + a_{-m} r^{-m}$,

be the proposed fraction of the kind considered: then, expressing each term as a vulgar fraction, we have

$$N = \frac{a_{-1}}{r} + \frac{a_{-2}}{r^2} + \frac{a_{-3}}{r^3} + \&c. + \frac{a_{-m}}{r^m}$$

$$= \frac{a_{-1} r^{m-1} + a_{-2} r^{m-2} + a_{-3} r^{m-3} + \&c. + a_{-m}}{r^m},$$

by reducing the individual terms to a common denominator: and from this we conclude that any quantity consisting of $m$ digits to the right of the separating point, may be represented by a vulgar fraction whose numerator is the said collection of digits considered integral, and denominator the $m^{\text{th}}$ power of the radix, or unity followed by $m$ zeros.

Ex. In the senary scale of notation, we shall have

$$.5324 = \frac{5}{6} + \frac{3}{6^2} + \frac{2}{6^3} + \frac{4}{6^4}$$

$$= \frac{5 . 6^3 + 3 . 6^2 + 2 . 6 + 4}{6^4}$$

$$= \frac{5324}{10000}.$$

If the base of the system be 11, we shall have likewise

$$.107t9 = \frac{1}{11} + \frac{0}{11^2} + \frac{7}{11^3} + \frac{10}{11^4} + \frac{9}{11^5}$$

$$= \frac{107t9}{11^5} = \frac{107t9}{100000}.$$

344. Cor. 1. Hence it follows conversely that in any scale of notation a fraction with unity followed by $m$ zeros for its denominator, may be expressed in the form of a whole number, by placing a point in the numerator so that it may have $m$ digits on its right hand.

Thus, $\left(\frac{23}{100}\right)_4 = (.23)_4$; $\left(\frac{25072}{1000}\right)_9 = (25.072)_9$,

$$\left(\frac{798}{100000}\right)_{11} = (.00798)_{11}, \text{ \&c.}$$

345. Cor. 2. In the addition and subtraction of fractions expressed after the manner of whole numbers, it is manifest that in order to have the same denominations of units combined together, the points in all the quantities concerned must be in the same vertical line, and then the operation may be effected as in integral quantities: but some additional considerations will be necessary to estimate properly the results of the operations of multiplication and division.

Let $P$ and $Q$ contain $p$ and $q$ digits to the right of the separating point respectively; then may these quantities be represented as vulgar fractions by $\frac{P}{r^p}$ and $\frac{Q}{r^q}$:

$$\text{whence their product} = \frac{P}{r^p} \times \frac{Q}{r^q} = \frac{PQ}{r^{p+q}},$$

which has therefore $p+q$ digits to the right of the separating point:

$$\text{also their quotient} = \frac{P}{r^p} \div \frac{Q}{r^q} = \frac{\left(\frac{P}{Q}\right)}{r^{p-q}},$$

which has therefore $p-q$ digits to the right of the said distinguishing mark. Similar processes may be adopted in their involution and evolution.

346. Cor. 3. To convert to the form of a whole number, a vulgar fraction whose denominator is *not* unity followed by a number of zeros, the same principle combined with the ordinary operation of division will suffice.

Thus, in the scale whose radix is 7, we have

$$\frac{54}{300} = \tfrac{1}{3}\left(\frac{54}{100}\right) = \tfrac{1}{3}\,(.54) = .16:$$

so, likewise in the nonary scale, we shall find

$$\frac{258}{527} = \frac{1}{527}\,(258.00) = .45: \ \&c;$$

whence we infer that any fraction whatever may be represented after the manner of whole numbers, by affixing to the numerator as many cyphers as may be necessary, and then effecting the division of this result by the denominator.

The zeros may be affixed to the numerator as it stands; or, when the division has been effected as far as the place of units, the remainder being then less than the divisor, if we affix a cypher to it, we reduce it to units of the next inferior order, and this will also be the denomination of the figure then obtained for the quotient; and so on.

347. Cor. 4. Should the division made in the manner above prescribed never terminate, but the terms continually reproduce one another in the same order, the quotient is termed *periodical*, the figures which recur being styled its *Period:* and the quantity is denominated a *simple* or *mixed* periodical fraction according as the period commences at the first digit on the right of the separating point, or afterwards.

Thus, in the scale whose radix is 5, we have

$$\tfrac{1}{3} = \tfrac{1}{3}(1.000000\ \&c.) = .131313\ \&c.$$

which is a simple periodical quantity: and in the denary scale, we have

$$\frac{101}{110} = \frac{1}{11}\,(10.10000\ \&c.) = .91818\ \&c.$$

which is a mixed periodical fraction.

348. *Every periodical quantity may be expressed exactly by means of a Vulgar Fraction.*

First, taking a simple periodical quantity where each period consists of $q$ digits in a scale whose radix is $r$, let us assume

$$\Sigma = .QQQ \text{ \&c.};$$

$$\therefore \text{ by (342)}, \; r^q \Sigma = Q.QQQ \text{ \&c.},$$

whence, by equal subtraction, we obtain

$$(r^q - 1)\Sigma = Q \text{ and } \therefore \Sigma = \frac{Q}{r^q - 1}:$$

next let the quantity be mixedly periodical, in which $P$ and $Q$ consist of $p$ and $q$ digits respectively: and then let us make

$$\Sigma = .PQQQ \text{ \&c};$$

$$\therefore r^{p+q}\Sigma = PQ.QQQ \text{ \&c., and } r^p \Sigma = P.QQQ \text{ \&c}:$$

whence, taking the difference of these two expressions, we find

$$(r^{p+q} - r^p)\,\Sigma = PQ - P \text{ and } \Sigma = \frac{PQ - P}{r^p(r^q - 1)}.$$

349. *If the radix of the scale be given, the digits expressing any proposed fraction in that scale may be found.*

As before, taking the general formula for fractional quantities, we have

$$N = a_{-1}r^{-1} + a_{-2}r^{-2} + a_{-3}r^{-3} + \text{\&c.} + a_{-m}r^{-m}$$

$$= \frac{a_{-1}}{r} + \frac{a_{-2}}{r^2} + \frac{a_{-3}}{r^3} + \text{\&c.} + \frac{a_{-m}}{r^m}:$$

and from this it is required to determine the digits

$$a_{-1},\; a_{-2},\; a_{-3},\; \text{\&c.},\; a_{-m}:$$

$$\text{now}\quad r\,N = a_{-1} + \frac{a_{-2}}{r} + \frac{a_{-3}}{r^2} + \&\text{c.} + \frac{a_{-m}}{r^{m-1}}:$$

$$r^2 N = a_{-1} r + a_{-2} + \frac{a_{-3}}{r} + \&\text{c.} + \frac{a_{-m}}{r^{m-2}}:$$

$$r^3 N = a_{-1} r^2 + a_{-2} r + a_{-3} + \&\text{c.} + \frac{a_{-m}}{r^{m-3}}:$$

&c.........................................

which results demonstrate that the first, second, third, &c. digits, reckoned from the separating point, are the integers in the products which arise from multiplying the fractional parts of $N$, $rN$, $r^2N$, &c, successively by $r$.

Ex. 1. Let it be required to express .015625 in the octenary scale.

Here we have $N = \;.015625;$

$$8N = 0\,.\,125000, \;\therefore\; a_{-1} = 0,$$

$$8^2 N = 1\,.\,000000, \;\therefore\; a_{-2} = 1;$$

hence .015625 in the denary scale is equivalent to .01 in the octenary; and this may easily be verified, for, in the octenary scale we have

$$.\,01 = \frac{0}{8} + \frac{1}{8^2} = \frac{1}{64} = .015625.$$

Ex. 2. Transform 14.125 from the denary to the duodenary scale of notation.

First, for the integral part, we have as in (323),

$$\begin{array}{r|ll} 12 & 14 & \\ \hline 12 & 1 & 2 = a_0, \\ \hline & 0 & 1 = a_1; \end{array}$$

$\therefore$ 14 denary = 12 duodenary:

again, for the part which is fractional, we have

$$\begin{array}{r} .125 \\ 12 \\ \hline 1.500, \ \therefore\ a_{-1}=1, \\ 12 \\ \hline 6.000, \ \therefore\ a_{-2}=6; \end{array}$$

whence the required duodenary expression will be 12.16, the correctness of which may be shewn as before.

350. Cor. Hence any fraction or mixed quantity may be transformed from a scale whose radix is $r$, to another whose radix is $r'$, by first expressing it in the denary scale, and then transforming the result into the scale whose radix is $r'$.

351. We shall now proceed more particularly to apply the principles laid down in the preceding pages to what are termed *Decimal Fractions* or *Decimals*, and *Duodenary Arithmetic* or *Duodecimals*: to the former by investigating the rules by which all the operations upon them are performed, and to the latter by exhibiting the solutions of such instances as are most frequently met with in practice.

## III. Decimals.

352. Def. In the denary scale of notation we have seen that any number $N$ may be expressed by

$$a_m 10^m + a_{m-1} 10^{m-1} + a_{m-2} 10^{m-2} + \&c. + a_2 10^2 + a_1 10^1 + a_0 10^0,$$

whenever it is an integral quantity; and in this the local value of each digit increases in a tenfold proportion from the place of units: wherefore if we continue the terms of this formula to the right from the same place, so that we have

$$N = a_m 10^m + a_{m-1} 10^{m-1} + \&c. + a_1 10^1 + a_0 10^0 + a_{-1} 10^{-1} + a_{-2} 10^{-2} + \&c.,$$

it is obvious that

$$a_m 10^m + a_{m-1} 10^{m-1} + \&c. + a_1 10^1 + a_0 10^0$$

representing whole numbers, the remaining part of the expression

$$a_{-1} 10^{-1} + a_{-2} 10^{-2} + \&c. \text{ or } \frac{a_{-1}}{10} + \frac{a_{-2}}{10^2} + \&c.$$

will be *Decimal Fractions*, so called because the denominators are all powers of 10; and the local value of every digit towards the right hand manifestly decreases in a tenfold ratio.

Thus, if the integral and fractional parts of the expression be, as before, separated by a point, and $a_{-1}$, $a_{-2}$, $a_{-3}$, &c. be taken to represent the digits 1, 2, 3, &c., we shall have

$$\frac{a_{-1}}{10} + \frac{a_{-2}}{10^2} + \frac{a_{-3}}{10^3} + \&c. = .123 \&c.;$$

wherein it is evident that 1 denotes one tenth, 2 two hundredths, 3 three thousandths, &c. of an unit, the local value of each digit being one tenth part of that which immediately precedes it.

353. Cor. Hence, decimal fractions, which are expressed as whole numbers with a point placed to their left hand, are in reality equivalent to vulgar fractions having 10 and its powers for denominators, so that if $P$ be a magnitude consisting of $p$ decimal places, its value expressed as a vulgar fraction is

$$\frac{P}{10^p}.$$

Thus, $.15 = \frac{1}{10} + \frac{5}{100} = \frac{15}{100}$; $.047 = \frac{4}{100} + \frac{7}{1000} = \frac{47}{1000}$;

$$3.026 = 3 + \frac{2}{100} + \frac{6}{1000} = \frac{3026}{1000};$$

$$.280 = \frac{2}{10} + \frac{8}{100} = \frac{28}{100} = .28, \ \&c.$$

It appears, therefore, that 0 placed after a decimal fraction does not alter its value; but placed before it, diminishes the value of each digit tenfold.

354. The reduction of decimals to common denominators, being equivalent to that of the fractions which represent them, will manifestly be effected by making the numbers of digits to the right of the decimal point the same in each: thus,

$$8\,.\,75 = 8 + \frac{7}{10} + \frac{5}{100} = \frac{875}{100} = \frac{8750}{1000},$$

$$6\,.\,273 = 6 + \frac{2}{10} + \frac{7}{100} + \frac{3}{1000} = \frac{6273}{1000};$$

$\therefore$ 8.75 and 6.273 so reduced, are equivalent to $\frac{8750}{1000}$ and $\frac{6273}{1000}$; which, by indicating the denominators by points, become 8.750 and 6.273 respectively.

In this manner, decimals are prepared for the operations of addition and subtraction.

Ex. 1. Required the sum and difference of 73.5083 and 29.327386.

| | | | |
|---|---|---|---|
| Here | 73.508300 | also | 73.508300 |
| and | 29.327386 | and | 29.327386 |

$\therefore$ the sum $=$ 102.835686, and the difference $=$ 44.180914:

and the following rule is universal:

Place the digits in such a manner that those of the same denomination may all be in a vertical line, and take the sum or difference as in whole numbers, the decimal point in the result being placed immediately under those in the quantities proposed.

Ex. 2. What is the product of $.287$ and $.305$?

Here we have $.287 = \dfrac{287}{1000}$ and $.305 = \dfrac{305}{1000}$;

$$\therefore \text{ their product} = \frac{287 \times 305}{1000000} = \frac{87535}{1000000} = .087535:$$

which has obviously as many places of decimals as are contained in both the multiplicand and multiplier together.

Ex. 3. Let us suppose $.P$ and $.Q$ to contain $p$ and $q$ decimals respectively; then, as has been shewn in (353),

$$.P = \frac{P}{10^p} \text{ and } .Q = \frac{Q}{10^q}:$$

$$\therefore \text{ the product } = .P \times .Q = \frac{P}{10^p} \times \frac{Q}{10^q} = \frac{PQ}{10^{p+q}}:$$

from which may be immediately deduced the general conclusion, that the multiplication of decimals is performed as in whole numbers, and that the product comprises as many decimal places as are found in both the multiplicand and multiplier.

Ex. 4. To find the product of two mixed quantities as $7.854$ and $23.82$, we have, however, no occasion to suppose them reduced to their corresponding vulgar fractions: for since the values of the digits in the units' place are always absolute values and not affected by their situation, the denominations of the results arising from multiplying by the digits in other situations, may be decided by them: thus,

$$\begin{array}{rr} \text{the multiplicand} = & 7.854 \\ \text{the multiplier} = & \underline{23.82} \\ & 157.08 \\ & 23.562 \\ & 6.2832 \\ & \underline{.15708} \end{array}$$

$\therefore$ the product $= 187.08228$; and in this operation the

multiplication by 3, which is in the units' place, is supposed to be first performed, and the other lines begin successively one or more places to the left or right of this, according as they are the results of multiplications by whole numbers or fractions.

Ex. 5. To divide 4.104 by 3.42, we have in vulgar fractions

$$4.104 = \frac{4104}{1000} \text{ and } 3.42 = \frac{342}{100};$$

$$\therefore \text{ the quotient } = \frac{4104}{1000} \div \frac{342}{100} = \frac{4104}{1000} \times \frac{100}{342}$$

$$= \frac{1}{1000} \times \frac{410400}{342} = \frac{1200}{1000} = 1.2:$$

and here having affixed to the dividend as many cyphers as we please, and performed the division as in integral quantities, we observe that the number of decimal places in the quotient is equal to the excess of the number in the dividend above that in the divisor.

Ex. 6. Generally, let the dividend $.P$ and divisor $.Q$ contain $p$ and $q$ decimal places respectively; then we have

$$.P = \frac{P}{10^p} \text{ and } .Q = \frac{Q}{10^q};$$

$\therefore$ the quotient

$$= \frac{P}{10^p} \div \frac{Q}{10^q} = \frac{P}{10^p} \times \frac{10^q}{Q} = \frac{P}{Q} \times \frac{1}{10^{p-q}}, \text{ or } \frac{P}{Q} \times 10^{q-p},$$

according as $p$ is greater or less than $q$:

first, let $p$ be greater than $q$; then we see that after the division is effected as in whole numbers, the quotient must comprise $p-q$ decimal places;

next, let $p$ be less than $q$, and it is then evident that we must affix to the quotient obtained as before, a number of cyphers $= q-p$, and the result will be whole numbers.

Ex. 7. If $.P$ represent a quantity consisting of $p$ decimals, which is to be raised to any power, then since $.P=\frac{P}{10^p}$, we have

$$\text{the square of } .P=\frac{P^2}{10^{2p}}, \text{ which has } 2p \text{ decimals;}$$

$$\text{the cube of } .P=\frac{P^3}{10^{3p}}, \ \ldots\ldots\ldots\ 3p \ \ldots\ldots\ldots,$$

and so on: and conversely, every quantity considered to be the square, cube, &c. of a decimal, must comprise a number of decimal places equal to some multiple of 2, 3, &c. respectively; and the numbers of decimal places in the roots will obviously be one half, one third, &c. of the numbers in the quantities proposed.

355. Cor. If we have two or more quantities, as $M$ and $N$, containing $m$ and $n$ decimals respectively, and $m'$ and $n'$ of such decimals only be taken into consideration, we may easily find the errors which will be occasioned in each of the operations by such a mode of proceeding.

Thus, by expressing the quantities proposed by means of vulgar fractions, the true values of $M$ and $N$ will be $\frac{M}{10^m}$ and $\frac{N}{10^n}$ respectively: and if $M'$ and $N'$ denote the first $m'$ and $n'$ digits of $M$ and $N$ respectively, we shall have the approximate values equivalent to

$$\frac{M'}{10^{m'}} \text{ and } \frac{N'}{10^{n'}};$$

$$\therefore \text{ the true sum } = \frac{M}{10^m}+\frac{N}{10^n},$$

$$\text{and the approximate sum } = \frac{M'}{10^{m'}}+\frac{N'}{10^{n'}}:$$

whence the error $= \frac{M}{10^m} + \frac{N}{10^n} - \frac{M'}{10^{m'}} - \frac{N'}{10^{n'}}$

$$= \frac{1}{10^{m'}}\left\{\frac{M}{10^{m-m'}} - M'\right\} + \frac{1}{10^{n'}}\left\{\frac{N}{10^{n-n'}} - N'\right\}:$$

similar steps may be taken whatever number of quantities be added together; and, in the case of subtraction, they will not be different.

Again, the true product $= \frac{MN}{10^{m+n}}$,

and the approximate product $= \frac{M'N'}{10^{m'+n'}}$:

whence the error

$$= \frac{MN}{10^{m+n}} - \frac{M'N'}{10^{m'+n'}} = \frac{1}{10^{m'+n'}}\left\{\frac{MN}{10^{m+n-m'-n'}} - M'N'\right\}$$

Also, the true quotient $= \frac{M}{10^{m-n}N}$,

and the approximate quotient $= \frac{M'}{10^{m'-n'}N'}$;

the quantities $m$ and $m'$ being considered respectively greater than $n$ and $n'$:

wherefore the error

$$= \frac{M}{10^{m-n}N} - \frac{M'}{10^{m'-n'}N'} = \frac{1}{10^{m'-n'}}\left\{\frac{M}{10^{m-n-m'+n'}N} - \frac{M'}{N'}\right\}.$$

Similarly, for the operations of Involution and Evolution upon either of the quantities $M$ and $N$.

356. Def. *Circulating* or *Recurring* decimals are those wherein the same figure or set of figures is *indefinitely* repeated: the quantity repeated is styled a simple or compound *Repetend*, according as it consists of one, or two or more figures, as *.333* &c., *.232323* &c., *.135135* &c., and it is sometimes distinguished by means of a simple point or dot; thus, $.\dot{3}$ &c., $.\dot{2}\dot{3}$ &c., $.\dot{1}3\dot{5}$ &c., are equivalent to the quantities above written.

Any finite decimal may be considered as a recurring one having 0 for its repetend.

If the decimal consist of a part which does not recur, it is called a *mixed* circulating decimal.

357. Cor. Circulating or recurring decimals are therefore equivalent to quantities forming indefinitely extended geometrical progressions, having either $\frac{1}{10}$, or some powers of it, for their common ratios: thus, in the instances above given,

$$.333 \text{ \&c.} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \text{\&c. } \textit{in infinitum:}$$

$$.232323 \text{ \&c.} = \frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \text{\&c. } \textit{in infinitum:}$$

$$.135135135 \text{ \&c.} = \frac{135}{10^3} + \frac{135}{10^6} + \frac{135}{10^9} + \text{\&c. } \textit{in infinitum:}$$

and their treatment will consequently be reduced to the management of the sums of such geometrical series, as found by article (270) in a preceding Chapter.

358. *To express the value of a recurring decimal by means of a Vulgar Fraction.*

Here it is obvious that we have merely to find the sum of the infinitely continued geometrical series, which is equivalent to the proposed recurring decimal, by means of the formula

$$\Sigma = \frac{a}{1-r};$$

and the method adopted in the following examples manifestly amounts to the same thing.

Ex. 1. To find the value of .999 &c. *in infinitum*.

Assume $\Sigma = .9999$ &c. *in infinitum;*

$\therefore\ 10\Sigma = 9.9999$ &c. *in infinitum:*

$\therefore\ 10\Sigma - \Sigma$ or $9\Sigma = 9$, and $\therefore\ \Sigma = 1$, the value required.

Ex. 2. Required the value of .2525 &c. *in infinitum.*

Let $\Sigma = .2525$ &c.; $\therefore$ $100\Sigma = 25.2525$ &c.:

whence by subtraction, we have $99\Sigma = 25$, and $\therefore$ $\Sigma = \frac{25}{99}$: which may be verified by actual division.

Ex. 3. What is the value of $.PPP$ &c. *in infinitum,* where $P$ contains $p$ digits?

If $\Sigma = .PPP$ &c., then will $10^p \Sigma = P.PPP$ &c.;

$$\therefore (10^p - 1)\Sigma = P, \text{ and } \Sigma = \frac{P}{10^p - 1}:$$

that is, the equivalent fraction has the repetend of the proposed quantity for its numerator, and as many 9's as there are figures in it, for its denominator.

Conversely, $\frac{P}{10^p - 1}$ is convertible into a circulating decimal, wherein $p$ digits recur.

359. If the decimal contain other figures prefixed to those which recur, its value may be expressed fractionally by an extension of the same process.

Ex. 1. What is the value of .1666 &c. *in infinitum?*

Assume $\Sigma = .1666$ &c. *in infinitum:*

then $100\Sigma = 16.666$ &c. and $10\Sigma = 1.666$ &c.;

$$\text{wherefore } 90\Sigma = 15, \text{ and thence } \Sigma = \frac{15}{90} = \frac{1}{6}.$$

Ex. 2. Let the decimal proposed be .7485353 &c.;

$$\text{then } 10^5\Sigma = 74853.5353 \text{ \&c.},$$

$$\text{and } 10^3\Sigma = \quad 748.5353 \text{ \&c.};$$

$$\therefore 99000\Sigma = 74105, \text{ and } \Sigma = \frac{74105}{99000} = \frac{14821}{19800}.$$

Ex. 3. Generally, if the decimal be $.PQQQ$ &c. wherein $P$ and $Q$ contain $p$ and $q$ digits respectively;

$$\text{let } \Sigma = .PQQQ \text{ \&c.};$$

$$\therefore\ 10^{p+q}\Sigma = PQ\,.\,QQQ \text{ \&c. and } 10^{p}\Sigma = P.QQQ \text{ \&c.};$$

$$\therefore\ (10^{p+q} - 10^{p})\,\Sigma = PQ - P,$$

$$\text{and } \Sigma = \frac{PQ-P}{10^{p+q}-10^{p}} = \frac{PQ-P}{10^{p}(10^{q}-1)};$$

which is a general theorem, wherein $P$ denotes the number that does not recur, and $PQ$ the non-recurring and recurring parts together.

Hence, conversely, a fraction, of the form $\dfrac{PQ-P}{10^{p}(10^{q}-1)}$, will be convertible into a mixed circulating decimal, wherein $p$ and $q$ are the numbers of digits in the non-recurring and recurring parts respectively.

360. The last two articles and the examples appended to them, enable us to express the values of all recurring decimals in fractional forms; and the addition, subtraction, &c. of such quantities will be effected by the performance of the same operations upon their fractional representatives.

Ex. 1. Required the sum, difference, product and quotient of the circulating decimals .999 &c. and .1666 &c.

$$\text{Here, the sum} = .999 \text{ \&c.} + .1666 \text{ \&c.} = 1 + \frac{1}{6} = \frac{7}{6} = 1.1666 \text{ \&c.}:$$

$$\text{the difference} = .999 \text{ \&c.} - .1666 \text{ \&c.} = 1 - \frac{1}{6} = \frac{5}{6} = .8333 \text{ \&c.}:$$

$$\text{the product} = .999 \text{ \&c.} \times .1666 \text{ \&c.} = 1 \times \frac{1}{6} = \frac{1}{6} = .1666 \text{ \&c.}:$$

$$\text{the quotient} = .999 \text{ \&c.} \div .1666 \text{ \&c.} = 1 \div \frac{1}{6} = 6 = 6.000 \text{ \&c.}$$

Ex. 2. What are the square, cube, &c. of .333 &c.?

Here, by (358), we have .333 &c. $= \frac{3}{9} = \frac{1}{3}$:

$\therefore$ the square $= \left(\frac{1}{3}\right)^2 = \frac{1}{9} = .111$ &c.;

the cube $= \left(\frac{1}{3}\right)^3 = \frac{1}{27} = .037037$ &c.;

the fourth power $= .012345679012345679$ &c.; and so on.

Ex. 3. Required the square root of .02777 &c.

By (359), we have .02777 &c. $= \frac{1}{36}$; and therefore the required root $= \frac{1}{6} = .1666$ &c.: and similarly of other roots.

361. *Every vulgar fraction in its lowest terms may be converted into a recurring decimal, except it be of the form*

$$\frac{a}{2^p 5^q}.$$

For, the fraction being in its lowest terms, and the only factors of 10 and its powers being 2 and 5 and their powers, it is obvious that if cyphers be affixed to the numerator, the result of the division by the denominator can never terminate, unless that denominator be composed of powers of one or both of the numbers 2 and 5: or, in other words, unless the fraction be of the form

$$\frac{a}{2^p 5^q}.$$

362. Cor. 1. Hence every fraction of the form $\frac{a}{2^p 5^q}$ may be expressed by a terminating decimal; and it will consist of a number of decimal places equal to the greater of the quantities $p$ and $q$.

For, if $p$ be $> q$, $\dfrac{a}{2^p 5^q} = \dfrac{a5^{-q}}{2^p} = \dfrac{a5^{p-q}}{2^p 5^p} = \dfrac{a5^{p-q}}{10^p}$, which therefore comprises $p$ decimal places:

and if $p$ be $< q$, $\dfrac{a}{2^p 5^q} = \dfrac{a2^{-p}}{5^q} = \dfrac{a2^{q-p}}{2^q 5^q} = \dfrac{a2^{q-p}}{10^q}$, which therefore consists of $q$ decimals.

Ex. Thus, $\dfrac{23}{20} = \dfrac{23}{2^2 . 5} = 1.15$, which has two decimal places; and $\dfrac{143}{250} = \dfrac{143}{2 . 5^3} = .572$, comprising three places of decimals.

363. Cor. 2. Every fraction in its lowest terms, whose denominator is not of the form $2^p 5^q$, being convertible into a recurring decimal, the number of the figures which recur will always be less than its denominator.

For, in performing the division by the denominator, it is evident that the remainder at every step must be less than the divisor; and, therefore, that one or more of the remainders will recur before a number of digits has been obtained for the quotient equal to the number of units in the divisor.

Ex. Thus, $\dfrac{2}{3} = .\dot{6}66$ &c., where the repetend consists of one digit:

also, $\dfrac{5}{7} = .\dot{7}1428\dot{5}714285$ &c., in which the repetend consists of six figures.

364. Cor. 3. Since, by (362), $\dfrac{a}{2^p 5^q}$ may always be expressed by a terminating decimal of $p$ or $q$ places, according as $p$ is greater or less than $q$, it follows that, if $a$ and $b$ have no common measure greater than 1, the fraction $\dfrac{a}{2^p 5^q b}$ will

be convertible into a mixed circulating decimal, in which the greater of the indices $p$ and $q$ will be the number of decimals in the part which does not recur.

Ex. The fraction $\frac{7}{180}$, which is equivalent to $\frac{7}{2^2.5.9}$, converted into a decimal, becomes $.03\dot{8}88$ &c.; wherein the number of digits that do not recur is obviously 2.

365. Cor. 4. On the same hypothesis, the product of

$$\frac{a}{2^p 5^q b} \text{ and } \frac{c}{2^r 5^s d} = \frac{ac5^{p-q+r-s}}{bd10^{p+r}},$$

which will give a mixed quotient obviously containing $p+r$ decimals that do not recur, if $p+r$ be greater than $q+s$, and $\frac{ac}{bd}$ be a fraction in its lowest terms; and the contrary.

Again, their quotient $= \frac{ad5^{p-q-r+s}}{bc10^{p-r}}$, which, on a similar hypothesis, will produce a mixed quantity wherein there are $p-r$ decimals that do not recur: and so on, of their powers, &c.

Ex. Let the fractions be $\frac{3}{280}$ and $\frac{13}{1100}$, which are equivalent to $\frac{3}{2^3.5.7}$ and $\frac{13}{2^2.5^2.11}$ respectively: then their product is $\frac{39}{308000}$, which expressed decimally becomes

$$.0001266233\dot{7}66233\dot{7} \text{ \&c.};$$

wherein $5=3+2$ of the decimals do not recur: and their quotient is $\frac{165}{182} = .906593\dot{4}06593\dot{4}$ &c., comprising $1=3-2$ non-recurring decimals.

## IV. Duodecimals.

366. Def. In the *Duodenary* or *Duodecimal* scale of notation whereof the radix is 12, the local values of figures increase in a twelvefold proportion from the units' place towards the left hand, and decrease in the same proportion from that place towards the right, analogously to what has been said respecting whole numbers and fractions in the common or decimal scale: and there belong to it twelve different digits,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ t,\ u,$$

the letters $t$ and $u$ as before denoting 10 and 11 respectively.

In the same manner, therefore, as we regard the number 10 in the common scale of notation, the number 12 must be referred to in this: or, in other words, we must carry one at every *twelve*, and borrow *twelve* instead of ten, &c., whenever the operations require it.

This will be best illustrated by the examples to which the scale is usually applied.

Ex. 1. Find the sum and difference of 75 feet 8 inches 7 parts, and 34 feet 10 inches and 4 parts.

First, we observe that, feet, inches, parts, seconds, &c. being in order connected by the invariable factor 12, they may all, except the first, be considered to be expressed in the duodenary scale of notation; and we must therefore transform the common numbers 75 and 34 into the same scale: thus,

$$\begin{array}{rll} 12) & 75 & \\ \hline 12) & 6 & 3 = a_0, \\ \hline & 0 & 6 = a_1; \\ \hline \end{array} \qquad \begin{array}{rll} 12) & 34 & \\ \hline 12) & 2 & t = b_0, \\ \hline & 0 & 2 = b_1; \\ \hline \end{array}$$

whence the two proposed quantities expressed *duodecimally*, are $63.87$ and $2t.t4$, the points separating the whole numbers from the fractions, as in decimals;

$\therefore$ the sum $= 63.87 + 2t.t4 = 92.6u$;

but, since here $92 = 9.12 + 2 = 110$, the required sum expressed in feet, inches and parts, is 110 feet 6 inches and 11 parts.

Again, their difference $= 63.87 - 2t.t4 = 34.t3$, which, in like manner $= 40$ feet 10 inches and 3 parts.

Ex. 2. Required the product of 9 feet 8 inches 7 parts, and 3 feet 10 inches.

$$\begin{array}{rr}
\text{Here the multiplicand} = & 9.87 \\
\text{and the multiplier} = & 3.t \\
\hline
 & 811t \\
 & 2519 \\
\hline
\end{array}$$

$\therefore$ the product $= 31.2tt = 37$ feet 2 inches 10 parts and 10 seconds, since 31 duodenary $= 37$ in the common scale.

Similarly of the involution of quantities of the same denominations.

Ex. 3. Divide 402 feet 5 inches and 2 parts by 25 feet 5 inches.

First, 402 feet 5 inches 2 parts $= 296.52$ duodenary:

and 25 feet 5 inches $= 21.\ 5$ ............

whence $21.5)\ 296.52\ (13.t = 15$ feet 10 inches.

$$\begin{array}{r}
215 \\
\hline
815 \\
643 \\
\hline
1922 \\
1922 \\
\hline
\end{array}$$

Ex. 4. Required the square root of 763 feet 1 inch 8 parts and 3 seconds.

First, 763 feet 1 inch 8 parts 3 seconds expressed in the duodenary scale, becomes 537.183; wherefore, pointing as in common numbers according to (338), we have

$$
\begin{array}{rl}
 & \dot{5}3\dot{7}.\dot{1}8\dot{3}0\ (23.76 = 27 \text{ feet } 7 \text{ inches and } 6 \text{ parts.} \\
 & 4 \\
\hline
43) & 137 \\
 & 109 \\
\hline
467) & 2t18 \\
 & 27t1 \\
\hline
4726) & 23730 \\
 & 23730 \\
\hline
\end{array}
$$

If the linear dimensions of rectangles and squares, of rectangular parallelopipeds and cubes &c. be expressed in feet, inches, parts, &c., their superficial and solid contents may be found by processes similar to what have been adopted in these examples and conversely.

367. With respect to the advantages and disadvantages of the various scales of notation which originate by assigning different values to $r$, it may be remarked, that it would obviously be desirable in point of practical convenience to select one wherein the number of figures expressing any given numerical magnitude, might be confined within limits not very extensive. This would further prevent excessive prolixity in the execution of the arithmetical operations: and, by practice, it soon becomes equally easy to perform these operations in any scale, provided its radix be not a very large number.

As by (362) all terminating decimals in the common scale are comprehended in the form $\frac{a}{2^p 5^q}$; so, in the senary scale,

for instance, all such quantities would be comprised in the form $\frac{a}{2^p 3^q}$: but, within given limits, there are evidently more multiples of 3 and its powers than there are of 5 and its powers, and therefore the senary scale would appear to possess an advantage over the denary, at least in the expression of fractional quantities. Similar remarks will be applicable to the duodenary scale of notation.

The selection of the scale in common use was therefore probably not made from a comparison of its merits with those of other systems, but from some accidental circumstance, which is now generally supposed to have been that of computation among mankind having been first conducted by means of the fingers of both hands; and hence the name of *Digits* has been given to the figures in common use.

For a short account of this subject, as also of the notations of the *Greeks* and *Hebrews*, the reader is referred to the article *Notation* in Barlow's *Mathematical* and *Philosophical Dictionary*.

# CHAP. XII.

*On the different Forms and Kinds of Numbers, and the Solution of certain Arithmetical Problems dependent upon some of their simple Properties.*

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## I. General Forms of Numbers.

368. Def. General *Forms of Numbers* are certain Algebraical formulæ, which, by assigning successive values to one or more of the letters contained in them, produce in order all numbers whatsoever.

369. *If* M *be assumed to represent any number whatever, then may every whole number, however small or great, be expressed by one or other of the terms of the series*

$$Mm,\ Mm+1,\ Mm+2,\ Mm+3,\ \&c.,\ Mm+(M-1),$$

*by assigning a proper value to* m.

For, every number whatsoever must either be exactly divisible by $M$, or must leave for a remainder one or other of the numbers

$$1,\ 2,\ 3,\ \&c.,\ (M-1);$$

and therefore if a proper value be given to $m$, it manifestly follows that every whole number will be comprised in the above mentioned series.

The quantity $M$ which characterises any particular set of forms, is termed the *Modulus*, and its magnitude may be assumed at pleasure.

370. Cor. 1. If we give to $M$, the values 1, 2, 3, &c. in succession, we shall have the following corresponding general formulæ:

| Modulus. | General Forms of Numbers. |
|---|---|
| 1 | $m$; |
| 2 | $2m$, $2m+1$; |
| 3 | $3m$, $3m+1$, $3m+2$; |
| 4 | $4m$, $4m+1$, $4m+2$, $4m+3$; |
| &c. | &c.................................. |

and in each of these sets if $m$ be made equal to 0, 1, 2, 3, &c. in order, we shall obtain all numbers whatever.

Thus, to the modulus 4,

if $m=0$, we get 0, 1, 2, 3;

if $m=1$, ....... 4, 5, 6, 7;

if $m=2$, ....... 8, 9, 10, 11;

&c............................

and similarly of the other forms: and it may be observed that the number of different forms belonging to any modulus will always be equal to the number of units in that modulus.

371. Cor. 2. Hence if we wish to express any given number $N$ by means of the given modulus $M$, we have only to divide the former by the latter and to note the quotient $m$ and the remainder $R$, for then we shall manifestly have

$$N = Mm + R.$$

Ex. Represent 257 by means of the moduli 6, 11 and 13.

First, $6\,)\,257$ $\quad$ $11\,)\,257$ $\quad$ $13\,)\,257$

$\quad\quad 42\ \ 5,$ $\quad$ $23\ \ 4,$ $\quad$ $19\ \ 10;$

whence we have $257 = 6.42 + 5 = 11.23 + 4 = 13.19 + 10$.

372. Cor. 3. The number of forms, belonging to any given modulus, may frequently be exhibited in an abbreviated shape by the change of an algebraical sign.

Thus, to the modulus 3, we have for all numbers whatever the three following forms

$$3m,\ 3m+1 \text{ and } 3m+2;$$

but since $3m+2=3(m+1)-1=3m'-1$, if $m'=m+1$, it is obvious that all numbers are comprised in the forms

$$3m \text{ and } 3m \pm 1.$$

Again, to the modulus 5, we have the five forms

$$5m,\ 5m+1,\ 5m+2,\ 5m+3 \text{ and } 5m+4,$$

which are likewise comprehended in the forms

$$5m,\ 5m \pm 1 \text{ and } 5m \pm 2.$$

And generally, to the modulus $M$, the forms similarly become $Mm$, $Mm \pm 1$, $Mm \pm 2$, $Mm \pm 3$, &c.

373. Before we proceed further, we will illustrate the use of these forms by applying them to the demonstration of a few arithmetical theorems.

Ex. 1. *The product of any two consecutive numbers is even.*

For, to the modulus 2, any two consecutive numbers may be expressed by $2m$ and $2m \pm 1$:

$$\therefore \text{ the product } = 2m(2m \pm 1) = 2(2m^2 \pm m),$$

which is divisible by 2, and therefore is an even number.

Hence, also, the continued product of any collection of consecutive numbers is even.

Ex. 2. *The product of any two odd numbers is odd, and that of any two even numbers is even.*

For, to the modulus 2, any two odd numbers $N$ and $N'$ may be expressed by $2m \pm 1$ and $2m' \pm 1$:

$$\therefore\ NN' = (2m \pm 1)(2m' \pm 1)$$
$$= 4mm' \pm 2(m + m') + 1$$
$$= 2\{2mm' \pm m \pm m'\} + 1,$$

which is of the form $2m + 1$, and therefore odd:

again, to the same modulus, any two even numbers $N$ and $N'$ may be represented by $2m$ and $2m'$:

$$\therefore\ NN' = 2m' \times 2m = 2(2mm'),$$

which is obviously of the form $2m$, and therefore even.

Hence, the continued product of any number of odd numbers is odd, and that of any number of even numbers is even; and the continued product of any number of odd and even numbers together is even.

Also, any power whatever of an odd number is odd, and of an even number even.

Ex. 3. *The product of any three consecutive numbers is divisible by* $1.2.3$ *or* 6.

For, every number may be expressed by one or other of the quantities $3m$, $3m + 1$ and $3m + 2$, $m$ being indeterminate:

first, let $N_1 = 3m$, $\therefore N_2 = 3m + 1$ and $N_3 = 3m + 2$:

and since, by the first example, $N_2 N_3$ has been proved to be divisible by 2, it follows that $N_1 N_2 N_3$ is divisible by $1.2.3$ or 6:

secondly, let $N_1 = 3m + 1$, $\therefore N_2 = 3m + 2$ and $N_3 = 3m + 3$ $= 3(m + 1)$; whence $N_1 N_2$ being divisible by 2, we shall obviously have $N_1 N_2 N_3$ divisible by $1.2.3$ or 6:

lastly, let $N_1 = 3m + 2$, $\therefore$ $N_2 = 3m + 3 = 3(m + 1)$ and $N_3 = 3(m + 1) + 1$; wherefore if $m$ be odd, $m + 1$ is even, and $\therefore$ $N_2$ is divisible by $1 . 2 . 3$: but if $m$ be even, both $N_1$ and $N_3$ are divisible by 2, so that in each case the product $N_1 N_2 N_3$ is divisible by $1 . 2 . 3$ or 6.

By a similar mode of proceeding it may be proved that $N_1 N_2 N_3 N_4$ is divisible by $1 . 2 . 3 . 4$ or 24; and generally that $N_1 N_2 N_3 N_4$ &c. $N_r$ is divisible by $1 . 2 . 3 . 4$. &c. $r$.

From this we may conclude that all the coefficients of the expansion of $(1 + x)^m$ are integral quantities, when $m$ is an integer, as has been proved in (192).

Ex. 4. *If* N *be any odd number, then will* $(N^2+3)(N^2+7)$ *be divisible by* 32.

For, if $N = 2m+1$, we shall readily have $(N^2+3)(N^2+7)$ $= (4m^2 + 4m + 4)(4m^2 + 4m + 8) = 16(m^2 + m + 1)(m^2 + m + 2)$, which is manifestly a multiple of 32, since the last factor $m^2 + m + 2$, or $m(m + 1) + 2$, by Ex. 1, is always even.

Ex. 5. *Every square number is of one of the forms* 5m *or* $5m \pm 1$.

For, every number $N$ is of one of the forms $5m$, $5m \pm 1$ and $5m \pm 2$;

$\therefore$ $N^2 = (5m)^2 = 25m^2 = 5(5m^2)$, which is of the form $5m$:

or $N^2 = (5m \pm 1)^2 = 25m^2 \pm 10m + 1 = 5(5m^2 \pm 2m) + 1$,

which is of the form $5m + 1$;

or $N^2 = (5m \pm 2)^2 = 25m^2 \pm 20m + 4 = 5(5m^2 \pm 4m + 1) - 1$,

which is of the form $5m - 1$.

Ex. 6. *The difference of the squares of any two odd numbers is divisible by* 8.

For, if $N$ and $N'$ represent any two odd numbers; then, to the modulus 2, we may have

$$N = 2m + 1 \text{ and } N' = 2m' + 1:$$

$$\text{wherefore } N^2 - N'^2 = (2m+1)^2 - (2m'+1)^2$$
$$= 4(m^2 - m'^2 + m - m')$$
$$= 4\{m(m+1) - m'(m'+1)\};$$

and the quantity within the brackets being, by Ex. 1, divisible by 2, it follows that $N^2 - N'^2$ is divisible by 8.

Ex. 7. *If* N *be any number not divisible by* 3, *and* p *any even number whatever, then will* $N^p + 2$ *be a multiple of* 3.

For, every number not divisible by 3 is comprehended in the formulæ

$$N = 3m \pm 1:$$

∴ since $p$ is even, we have by the binomial theorem

$$N^p = (3m)^p \pm p(3m)^{p-1} + \frac{p(p-1)}{1.2}(3m)^{p-2} \pm \&c. + 1;$$

$$\therefore\ N^p + 2 = (3m)^p \pm p(3m)^{p-1} + \frac{p(p-1)}{1.2}(3m)^{p-2} \pm \&c. + 3,$$

which is obviously a multiple of 3.

Ex. 8. *Every number and its cube when divided by* 6 *leave the same remainder.*

For, to the modulus 6, every number is comprised in the forms

$$6m,\ 6m \pm 1,\ 6m \pm 2,\ 6m \pm 3:$$

of which the cubes are

$$(6m)^3,\ (6m)^3 \pm 3(6m)^2 + 3(6m) \pm 1,$$

$$(6m)^3 \pm 6(6m)^2 + 12(6m) \pm 8,\ (6m)^3 \pm 9(6m)^2 + 27(6m) \pm 27;$$

and when divided by 6, they leave the remainders

$$0, \ \pm 1, \ \pm 2 \text{ and } \pm 3,$$

respectively: in other words, all cube numbers, with regard to the modulus 6, are of the same forms as their roots.

As in Example 5, it is easily proved that all cube numbers are of the forms $4m$ and $4m \pm 1$, or $7m$ and $7m \pm 1$, or $9m$ and $9m \pm 1$: that every fourth power is of one of the forms $5m$ and $5m + 1$; and that every fifth power terminates with the same digit as its root, or that the fifth powers of all numbers, with respect to the modulus 10, are of the same forms as the numbers themselves.

## II. Forms and Properties of Prime Numbers.

374. Def. *Prime Numbers* are those which have no divisors except unity and themselves, and therefore cannot be divided into any number of equal integral parts greater than unity; and they are thus distinguished from numbers that are *composite.*

Thus, 2, 3, 5, 7, 11, 13, 17, 19, &c. are prime numbers.

375. *Every prime number greater than 2, is of one of the forms* $4m \pm 1$.

For, to the modulus 4, every number may be expressed by

$$4m, \ 4m \pm 1 \text{ or } 4m \pm 2,$$

whereof the first and last being divisible by 2, cannot contain any prime number greater than 2, and consequently every prime number must be comprised in one of the remaining forms

$$4m \pm 1.$$

Ex. Prime numbers of the form $4m+1$ are 1, 5, 13, 17, &c.;

and of the form $4m-1$ are 3, 7, 11, 19, &c.

376. Cor. Precisely in the same manner, to the modulus 6, it may be proved that every prime number greater than 3 is of one of the forms $6m \pm 1$: and similar formulæ, but not of equal simplicity, may be deduced when other moduli are adopted.

377. The formulæ $4m \pm 1$ and $6m \pm 1$, which are capable of expressing all prime numbers whatever by assigning proper values to $m$, will, it is manifest from trial, comprise at the same time other numbers which are not so; and in fact, it is easily demonstrable that no algebraical formula whatever can express prime numbers only.

For, since, by (369), every number may be expressed by the formula

$$N = Mx + R,$$

wherein $M$ may be any whole number whatever; if we suppose that when $x = m$, there is obtained the prime number $N_1$, so that

$$N_1 = Mm + R,$$

then, when $x = m + nN_1$, we should, taking $N_n$ to denote the corresponding value of $N$, have

$$N_n = M(m + nN_1) + R = Mm + MnN_1 + R$$

$$= N_1 + MnN_1 = (1 + Mn)\, N_1,$$

which is a composite number: hence, since $x$ may represent any whole number whatever, and such a value may always be given to it as renders $N$ a composite number, it follows that there can exist no formula which contains prime numbers exclusively.

378. *The number of prime numbers is indefinitely great.*

For, if possible, let there be a limited number of primes $N_1$, $N_2$, &c., $N_n$, whereof $N_n$ is the greatest; then it is evident that their continued product

$$N_1 N_2. \text{ \&c. } N_n$$

is divisible by each of them: and consequently that

$$N_1 N_2. \text{ \&c. } N_n + 1$$

is not divisible by any one of them: wherefore, this number must either be a prime number itself, or be divisible by one which is greater than $N_n$; therefore, in neither case, is $N_n$ the greatest prime number; or, in other words, both the number and magnitudes of prime numbers are indefinitely great.

379. *To determine whether any proposed number is a prime or not.*

If a number $N$ be not a prime, it is evident that we may have

$$N = a\,b:$$

now, if $b = a$, then $N = a^2$ and $\sqrt{N} = a$, or $N$ is divisible by $\sqrt{N}$:

again, if $b < a$, it is obvious that $b$ is $< \sqrt{N}$, and therefore $N$ is divisible by a quantity less than $\sqrt{N}$:

also, if $b > a$, it is equally manifest that $a$ is $< \sqrt{N}$; whence, as before, $N$ is divisible by a quantity less than $\sqrt{N}$: and it evidently follows that these conclusions will not hold good, unless the number can be resolved into two or more factors: in other words, we have obtained a criterion which will enable us to ascertain whether a number is prime, which is the circumstance of its not being capable of division, by any number either equal to or less than its square root.

380. In the preceding articles of this section, we have been considering the forms, &c., of numbers which are absolutely prime, and it now remains to explain in what circumstances two or more numbers may be relatively so.

**Def.** Two or more numbers are said to be prime to one another, when they have no integral common measure greater than unity; as, for instance, 9 and 16 are prime to each other, though neither of them considered by itself is a prime number; and the same may be said of 12, 25 and 49; &c.

381. *If the product* ab *be divisible by* c, *and* b *and* c *be prime to each other, then will* c *be a divisor of* a.

For, since $b$ and $c$ are prime to each other, their common measure determined by the ordinary process must be 1: that is, we may have the following operation, $b$ being greater than $c$:

$$\begin{array}{l}
c)\ b\ (p \\
\underline{pc} \\
\quad d)\ c\ (q \\
\quad \underline{qd} \\
\qquad e)\ d\ (r \\
\qquad \underline{re} \\
\qquad\quad 1)\ e\ (e \\
\qquad\quad \underline{e}
\end{array}$$

$$\therefore \text{ we have } b = cp + d, \ c = dq + e \text{ and } d = er + 1:$$

$$\text{whence } ab = acp + ad, \ ac = adq + ae \text{ and } ad = aer + a:$$

$$\therefore ab - acp = ad, \ ac - adq = ae \text{ and } ad - aer = a:$$

thus, since $ab$ is divisible by $c$, we shall have $ad$ so divisible: $\therefore ac - adq$ or $ae$ will be divisible by $c$; and thence $ad - aer$,

or $a$, has $c$ for a divisor: and a similar proof will be applicable when $b$ is less than $c$.

382. Cor. From this proposition, it is manifest that if $c$ be prime to $b$ and greater than $a$, the product $ab$ is not divisible by $c$.

383. *If two numbers be prime to each other, they are the least quantities that can express their ratio.*

For, if possible, suppose $\frac{A}{B} = \frac{a}{b}$, where $a$ and $b$ are respectively less than $A$ and $B$: then will $a = \frac{bA}{B}$, and therefore $B$ is a divisor of $bA$: but since, by the hypothesis, $A$ and $B$ are prime to each other, we must have $B$ for a divisor of $b$; which is absurd, because $b$ is less than $B$:

whence it follows that the fraction $\frac{A}{B}$ is *irreducible* and cannot be expressed in simpler terms.

384. Cor. Hence, therefore, if we have the irreducible fraction $\frac{A}{B}$ equal to the reducible fraction $\frac{a}{b}$, the terms of the latter will be equimultiples of those of the former.

For, if $d$ be the greatest common measure of $a$ and $b$ so that $a = da'$ and $b = db'$,

$$\text{we shall have } \frac{a}{b} = \frac{a'}{b'} = \frac{A}{B}:$$

but, by the last article, $a' = A$ and $b' = B$; $\therefore$ $a = dA$ and $b = dB$, or $a$ and $b$ are equimultiples of $A$ and $B$.

385. *If two numbers be prime to each other, their sum or difference is prime to each of them.*

Let $a$ and $b$ represent the two numbers; and, if possible, suppose $a$ and $a \pm b$ to have the common measure $d$, such that

$$a = pd \text{ and } a \pm b = qd: \therefore \pm b = (q-p)\,d;$$

whence $a$ and $b$ have the common measure $d$, which is contrary to the hypothesis: similarly of $b$ and $a \pm b$.

386. Cor. Hence, by a similar process, it may be made to appear that $a+b$ and $a-b$ are either prime to one another, or have the common measure 2.

387. *If one number be prime to each of two others, it is also prime to their product.*

Let the number $a$ be prime to each of the numbers $b$ and $c$; and, if possible, let $a = pd$ and $bc = qd$: then since $b$ and $c$ are prime to $a$ or $pd$, they are each prime to $d$: also, since

$$bc = qd, \text{ we have } \frac{b}{d} = \frac{q}{c},$$

and therefore, by (384), $c$ is a multiple of $d$; that is, $a$ and $c$ have a common measure $d$, which is absurd; therefore $a$ is prime to $bc$.

388. Cor. 1. If a number $a$ be prime to each of the numbers $b$, $c$, $d$, &c., it will also be prime to their continued product.

For, since $a$ is prime to $b$ and $c$, it is prime to their product $bc$; also, since $a$ is prime to $bc$ and $d$, it is prime to their product $bcd$; and so on.

389. Cor. 2. If we suppose $b = c = d =$ &c., we shall conclude that when $a$ is prime to $b$, it is also prime to $b^2$, $b^3$, &c., $b^m$: or, in other words, that if $\frac{b}{a}$ be a fraction in its lowest terms, $\frac{b^2}{a}$, $\frac{b^3}{a}$, &c., $\frac{b^m}{a}$ are also fractions in their lowest terms.

390. Cor. 3. If $a$, $b$, $c$, &c. be each of them prime to $a'$, $b'$, $c'$, &c.; then, by (388), $a\ b\ c$ &c. is prime to each of the quantities $a'$, $b'$, $c'$, &c. and therefore to their product $a'b'c'$ &c., by the same article.

Also, if $a=b=c=$ &c. and $a'=b'=c'=$ &c., then will $a^2$, $a^3$, $a^4$, &c. be prime to $a'^2$, $a'^3$, $a'^4$, &c. respectively; and if $\frac{a}{a'}$ be an irreducible fraction, the same may be said of $\left(\frac{a}{a'}\right)^2$, $\left(\frac{a}{a'}\right)^3$, &c., $\left(\frac{a}{a'}\right)^m$, and also of $\frac{a^m}{a'^{m'}}$, &c.

391. The investigation of the forms and properties of prime numbers formerly engaged much of the attention of mathematicians, but their researches have not been attended with any very satisfactory results. It is quite foreign to the object of the present performance to enter upon the speculations of Fermat, Euler and others, respecting this subject, and we shall therefore merely introduce one or two properties to give the reader some idea of the nature of the discoveries which they made.

392. *If* A *represent any number whatever, and* a, b, c, &c. *denote all the numbers less than* 2A *which are prime to it, then will every prime number greater than the prime factors of* A *be comprised in one or other of the forms*

$$4\mathrm{A}m \pm \mathrm{a},\quad 4\mathrm{A}m \pm \mathrm{b},\quad 4\mathrm{A}m \pm \mathrm{c},\quad \&c.$$

For, any number when divided by $4A$, must necessarily leave for a remainder one or other of the quantities

$$0,\ \pm 1,\ \pm 2,\ \pm 3,\ \&c.\ 2A, \text{ as appears from (372):}$$

whence, omitting all such remainders as are not prime to $4A$, and retaining the rest as $a$, $b$, $c$, &c., we shall manifestly have all prime numbers greater than the prime factors of $A$ comprised in the forms

$$4Am \pm a,\quad 4Am \pm b,\quad 4Am \pm c,\quad \&c.$$

Ex. If $A=1$, then $a=1$, $b=0$, &c. and all prime numbers are contained in the forms $4m\pm 1$;

$$\text{if } A=2, \text{ then } a=1,\ b=3, \text{ and } c=0, \text{ \&c.};$$

whence all prime numbers greater than 2 are comprehended in the forms $8m\pm 1$ and $8m\pm 3$: and so on.

393. *If* a *and* b *be any two numbers prime to each other, each of the quantities* b, 2b, 3b, &c., (a − 1) b, *when divided by* a, *leaves a different positive remainder.*

For, if possible, let the two terms $mb$ and $nb$ have the same remainder, so that

$$mb=Ma+R \text{ and } nb=Na+R:$$

$$\text{then will } (m-n)\,b=(M-N)\,a,$$

$$\text{and therefore } M-N=\frac{(m-n)\,b}{a}, \text{ an integral quantity}:$$

but, since $m-n$ is less than $a$ and $b$ is prime to it, therefore, by (382), the latter member of this equation cannot be integral, and consequently no two terms can have the same positive remainder.

394. Cor. Since there are $a-1$ different remainders each less than $a$, it is obvious that these remainders include all the numbers 1, 2, 3, &c., $(a-1)$.

395. *If* m *be a prime number, the coefficient of every term of the expansion of* $(1+x)^m$, *except the first and last, is divisible by* m.

For, the coefficient of the $n^{\text{th}}$ term of the expansion has been proved in (188) to be

$$\frac{m\,(m-1)\,(m-2)\,.\,\&c.\,(m-n+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(n-1)};$$

and it has been shewn in (192), that all the coefficients are whole numbers when the index is such; therefore, since $m$ is not divisible by any of the factors of the denominator, it follows that

$$\frac{(m-1)\,(m-2).\&c.\,(m-n+2)}{1\,.\,2\,.\,3\,.\,\&c.\,(n-1)}$$

must, of itself, be a whole number; and consequently the coefficient of every term, except the first and last, which do not involve $m$, must be divisible by $m$ without a remainder: but the same conclusions do not follow when $m$ is composite.

396. *If* m *be a prime number, and* N *be not divisible by* m, *then will* $N^{m-1}-1$ *be divisible by* m.

For, by the last article, we have seen that

$(1+x)^m-(1+x^m)$ is divisible by $m$, whenever $x$ is integral;

therefore, assuming $1+x=N$, we shall have

$N^m-1-(N-1)^m$ divisible by $m$, which suppose $=mQ_1$,

so that $N^m-N=(N-1)^m-(N-1)+mQ_1$;

similarly, $(N-1)^m-(N-1)=(N-2)^m-(N-2)+mQ_2$;

$(N-2)^m-(N-2)=(N-3)^m-(N-3)+mQ_3$;

&c. ..............................................

and continuing this process, we obviously at length arrive at

$1^m-1=(N-N)^m-(N-N)+mQ_n$, or $0=mQ_n$;

whence, by addition, $N^m-N=m\{Q_1+Q_2+Q_3+\&c.+Q_{n-1}\}$,

and is therefore divisible by $m$:

but $N^m-N$ being $=N(N^{m-1}-1)$, whereof $N$ is prime to $m$, it remains only that $N^{m-1}-1$ is divisible by $m$.

397. Cor. 1. Since $N^m - N$ is divisible by $m$, it manifestly follows that $N^m$, when divided by $m$, leaves the same remainder as $N$ divided by $m$ leaves.

398. Cor. 2. Because all the numbers 1, 2, 3, 4, &c., $m-1$, are prime to $m$, each of them when substituted for $N$ will render the expression $\dfrac{N^{m-1}-1}{m} = n$, a whole number: and since $m-1$ is necessarily even, there will evidently be $m-1$ values of $N$ comprised between the magnitudes $-\frac{1}{2}m$ and $\frac{1}{2}m$, which answer the same condition; or $N$ may be any one of the quantities

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 4, \quad \&c., \quad \pm \tfrac{1}{2}(m-1).$$

399. Cor. 3. Having proved that $\dfrac{N^{m-1}-1}{m}$ is a whole number, as $n$, we shall obviously have $N^{m-1}$ of the form $mn+1$; and consequently every power of $N$ whose exponent increased by 1 is a prime number, will be of the form $mn$ or $mn+1$, according as $N$ is or is not divisible by $m$.

Thus, $N^2$ is of the form $3n$ or $3n+1$:

$N^4$.................. $5n$ or $5n+1$:

$N^6$.................. $7n$ or $7n+1$:

$N^{10}$.................. $11n$ or $11n+1$:

&c. ....................................

400. Cor. 4. Since $m-1$ is an even number, we shall have

$$N^{m-1} - 1 = \{N^{\frac{1}{2}(m-1)} + 1\}\{N^{\frac{1}{2}(m-1)} - 1\},$$

one of the latter factors of which must manifestly be divisible by $m$, and consequently

$$N^{\frac{1}{2}(m-1)} \text{ is of the form } mn \pm 1;$$

that is, every power of $N$, the double of whose exponent increased by 1 becomes a prime number, is of the form $mn$ or $mn \pm 1$, according as $N$ is divisible by $m$ or not.

Thus, $N^2$ is of the form $5n$ or $5n \pm 1$:

$N^3$.................. $7n$ or $7n \pm 1$:

$N^5$.................. $11n$ or $11n \pm 1$:

$N^6$.................. $13n$ or $13n \pm 1$:

&c. ......................................

The discovery of the singular Theorem demonstrated in (396), is due to M. PETER FERMAT, a celebrated French mathematician, who was born in the year 1590: and the no less remarkable one which follows was invented by Sir JOHN WILSON, who proceeded to the degree of Bachelor of Arts in 1761, and subsequently became Fellow of St. Peter's College, Cambridge, and one of the Judges of his Majesty's Court of Common Pleas: and its demonstration may most readily be effected by means of a formula which is not very easily deduced from the principles of common Algebra.

401. *If* m *be a prime number, then will the continued product* 1 . 2 . 3 . &c. (m − 1), *when augmented by* 1, *be divisible by* m.

By means of the *Differential Calculus*, or the *Calculus of Finite Differences*, it is easily demonstrated that

$$1 \,.\, 2 \,.\, 3 \,.\, \&c.\ n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1 \,.\, 2}(n-2)^n - \&c.$$

to $n$ terms, whatever whole number $n$ may be:

whence, if $n$ be assumed $= m - 1$, we shall have

$$1 \,.\, 2 \,.\, 3 \,.\, \&c.\ (m-1) = (m-1)^{m-1} - \frac{(m-1)}{1}(m-2)^{m-1}$$

$$+ \frac{(m-1)(m-2)}{1 \,.\, 2}(m-3)^{m-1} - \&c.\ \text{to } m-1 \text{ terms:}$$

but, since $m$ is prime to each of the quantities $m-1$, $m-2$, $m-3$, &c., we obtain from *Fermat's* theorem, the following results:

$$(m-1)^{m-1} = m\,Q_1 + 1\,;$$

$$(m-2)^{m-1} = m\,Q_2 + 1\,;$$

$$(m-3)^{m-1} = m\,Q_3 + 1\,;$$

&c. . . . . . . . . . . . . . . . . . . . . . .

$$\therefore\ 1\,.\,2\,.\,3\,.\,\&c.\,(m-1) = 1 - \frac{m-1}{1} + \frac{(m-1)\,(m-2)}{1\,.\,2} - \&c.$$

$$\text{to } m-1 \text{ terms} + m\left\{Q_1 - \frac{m-1}{1}\,Q_2 + \frac{(m-1)\,(m-2)}{1\,.\,2}\,Q_3 - \&c.\right\}:$$

but, since $m-1$ is necessarily an even number, we shall have

$$1 - \frac{m-1}{1} + \frac{(m-1)\,(m-2)}{1\,.\,2} - \&c. \text{ to } m-1 \text{ terms}$$

$$= (1-1)^{m-1} - 1 = -1\,:$$

$$\therefore\ 1.2.3.\&c.(m-1) = m\left\{Q_1 - \frac{m-1}{1}\,Q_2 + \frac{(m-1)\,(m-2)}{1\,.\,2}\,Q_3 - \&c.\right\} - 1;$$

whence

$$1.2.3.\&c.\,(m-1) + 1 = m\left\{Q_1 - \frac{m-1}{1}\,Q_2 + \frac{(m-1)(m-2)}{1.2}\,Q_3 - \&c.\right\},$$

which is obviously divisible by $m$, as appears from (192). p. 238

402. Cor. 1. Since $m-1$ is an even number, the continued product $1\,.\,2\,.\,3\,.\,\&c.\,(m-1)$ is equivalent to

$$1\,(m-1)\;2\,(m-2)\;3\,(m-3)\,.\,\&c.\,\left\{\tfrac{1}{2}(m-1)\right\}^2;$$

which, when divided by $m$, manifestly leaves the same remainder as

$$\pm\left\{1\,.\,2\,.\,3\,.\,\&c.\,\tfrac{1}{2}\,(m-1)\right\}^2,$$

wherein the upper or lower sign is applicable, according as $\frac{1}{2}(m-1)$ is even or odd, or according as $m$ is of the form $4n+1$ or $4n-1$: consequently, in the former case we shall have

$$\{1.2.3.\&c.\ \tfrac{1}{2}(m-1)\}^2+1, \text{ divisible by } m;$$

and in the latter

$$\{1.2.3.\&c.\ \tfrac{1}{2}(m-1)\}^2-1, \text{ divisible by } m.$$

403. Cor. 2. By means of the former of these results, it appears that every prime number of the form $4n+1$ will divide the sum of two squares without a remainder.

404. Cor. 3. Since, in the latter of the expressions deduced above, we have

$$\{1.2.3.\&c.\ \tfrac{1}{2}(m-1)\}^2-1$$
$$=\{1.2.3.\&c.\ \tfrac{1}{2}(m-1)+1\}\{1.2.3.\&c.\ \tfrac{1}{2}(m-1)-1\},$$

it manifestly follows, that when $m$ is a prime number of the form $4n-1$, it will divide one or other of these factors.

405. Cor. 4. Because $\dfrac{1.2.3.\&c.\ (m-1)+1}{m}$, is an integral quantity, whenever $m$ is a prime number, we shall also have

$$\frac{1.2.(m-2).3.\&c.\ (m-3)(m-1)+1}{m},$$

and $\therefore$ $\dfrac{1.2^2.3.\&c.(m-3)(m-1)-1}{m}$, an integral quantity.

similarly, $\dfrac{1.2^2.3^2.\&c.\ (m-4)(m-1)+1}{m}$, &c. may be proved to be integral.

*Sir J. Wilson's* theorem furnishes a criterion for deciding whether a proposed number be prime or not, but the magnitude to which the continued product soon rises, renders it of much less practical utility than the one given in (379).

## III. Resolution of Numbers into their Prime Factors.

406. If any number $N$ be divisible by the prime numbers 2, 3, 5, &c., $p$, $q$, $r$, &c. times respectively in succession, it is obvious that $N = 2^p 3^q 5^r$ &c.; and if the primes be represented by $a$, $b$, $c$, &c., we shall have the more general formula

$$N = a^p b^q c^r \text{ \&c.}$$

407. *If a number be reducible to the form* $\mathrm{N} = \mathrm{a^p b^q c^r}$ &c., *where* a, b, c, &c. *are prime numbers, it will have* $(\mathrm{p}+1)$ $(\mathrm{q}+1)$ $(\mathrm{r}+1)$ &c. *different divisors.*

For $N$ is manifestly divisible by every term of each of the series

$$1,\ a,\ a^2,\ a^3,\ \text{\&c.},\ a^p;\ 1,\ b,\ b^2,\ b^3,\ \text{\&c.},\ b^q;\ 1,\ c,\ c^2,\ c^3,\ \text{\&c.},\ c^r;\ \text{\&c.}$$

which are $p+1$, $q+1$, $r+1$, &c. respectively in number:

therefore it is divisible by every combination of the terms of these series or by every term of their product; and, by (307), the number of terms in this product being

$$(p+1)\ (q+1)\ (r+1)\ \text{\&c.},$$

it follows that the number of divisors of $N$ is expressed by the same quantity.

It is also obvious, from the manner in which they are formed, that all the terms in this product are different from each other.

Ex. To find the number of divisors of 252.

| | | | |
|---|---|---|---|
| First, | 2) 252 | 3) 63 | 7) 7 |
| | 2) 126 | 3) 21 | 1 |
| | 63 | 7 | |

therefore, 252 resolved into its prime factors $= 2^2 . 3^2 . 7$; whence the number of its divisors, itself being considered one of them, is

$$(2+1)(2+1)(1+1) = 18.$$

408. Cor. 1. Conversely, to find a number having a given number $n$ of divisors, resolve it into the factors $x$, $y$, $z$, &c.: then, if we assume

$$x\,y\,z \text{ \&c.} = (p+1)(q+1)(r+1) \text{ \&c.},$$

the number required will be expressed by

$$a^p b^q c^r \text{ \&c. or } a^{x-1} b^{y-1} c^{z-1} \text{ \&c.}:$$

and it will obviously be the least when $a$, $b$, $c$, &c. are equivalent to 2, 3, 5, &c. and the indices are taken in the order of their magnitudes, beginning with the greatest.

Ex. Required a number having 15 divisors.

Here, $15 = 3 . 5 = xy$; whence the number will be expressed generally by $a^2 b^4$; and if $a = 2$ and $b = 3$, the number 324 has exactly 15 divisors, itself being considered one.

409. Cor. 2. Hence we may find a multiplier which will render any number a complete $m^{\text{th}}$ power.

For, if $N = a^p b^q c^r$ &c. be any proposed number, let the required multiplier be $P = a^x b^y c^z$ &c., so that

$$NP = a^{p+x} b^{q+y} c^{r+z} \text{ \&c.}$$

may be a perfect $m^{\text{th}}$ power: then it is evident that each of the indices $p+x$, $q+y$, $r+z$, &c. must either be equal to $m$ or to some multiples of it, as $am$, $\beta m$, $\gamma m$, &c.; whence we have

$$x = am - p, \quad y = \beta m - q, \quad z = \gamma m - r, \text{ \&c.},$$

and therefore $P = a^{\alpha m - p} b^{\beta m - q} c^{\gamma m - r}$ &c.: and the least number which will answer the purpose, will correspond to the least values of $a$, $\beta$, $\gamma$, &c. which render the indices positive.

Ex. Required a multiplier which will render 63 a perfect cube.

Here, $63 = 3 \cdot 21 = 3 \cdot 3 \cdot 7$; whence the required multiplier $= 3 \cdot 7^2$, and the complete cube $= 3^3 \cdot 7^3 = 21^3 = 9261$.

410. Cor. 3. The sum of all the divisors of $N$ will manifestly be equal to the sum of all the terms in the continued product

$$(1 + a + a^2 + \&c. + a^p)\,(1 + b + b^2 + \&c. + b^q)$$
$$(1 + c + c^2 + \&c. + c^r)\ \&c.$$
$$= \left(\frac{a^{p+1} - 1}{a - 1}\right)\left(\frac{b^{q+1} - 1}{b - 1}\right)\left(\frac{c^{r+1} - 1}{c - 1}\right)\ \&c.$$

411. Cor. 4. The number of divisors of any number is odd or even according as it is a square or not.

As before, if $N = a^p b^q c^r$ &c., the number of different divisors will be

$$(p + 1)\,(q + 1)\,(r + 1)\ \&c.;$$

but if $N$ be a square number, it is obvious that the indices $p$, $q$, $r$, &c. are all even numbers, and therefore the continued product $(p + 1)\,(q + 1)\,(r + 1)$ &c. must necessarily be odd:

when $N$ is not a square, one at least of the indices $p$, $q$, $r$, &c. must be an odd number, and therefore the continued product $(p + 1)\,(q + 1)\,(r + 1)$ &c. will in this case be even.

Hence, the converse is also true.

412. Cor. 5. Hence, the number of different ways into which $N = a^p b^q c^r$ &c. can be resolved into two factors, will be

$$= \tfrac{1}{2}\,(p + 1)\,(q + 1)\,(r + 1)\ \&c.;$$

because every divisor has another corresponding to it, such that their product $= N$: and if $N$ be not a square, this will be an integer, since then the continued product is even: should, however, $N$ be a square, and therefore, the continued product be odd, the number of different ways will be

$$\tfrac{1}{2}\left\{(p + 1)\,(q + 1)\,(r + 1)\ \&c. + 1\right\},$$

because then two factors are equal, and have been reckoned as only one.

413. Cor. 6. If the number of different ways in which $N$ may be resolved into two factors prime to each other be required, it is evident that if $m$ be the number of the primes $a$, $b$, $c$, &c. employed, we have only to make $p=q=r=$&c. to $m$ terms $=1$, and the required number corresponding will be

$$=\tfrac{1}{2}\{(1+1)\ (1+1)\ (1+1)\ \text{\&c. to } m \text{ factors}\}=2^{m-1}.$$

414. *If* $N=a^p b^q c^r$ &c., *then will the number of integers less than* N, *and prime to it, be expressed by*

$$(a-1)\ (b-1)\ (c-1)\ \text{\&c. } a^{p-1}b^{q-1}c^{r-1}\ \text{\&c.};$$

$$\text{or }\ N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right)\ \text{\&c.}$$

First, if we suppose $q=r=$&c.$=0$, and $\therefore N=a^p$, we shall manifestly have the $a^p$ whole numbers 1, 2, 3, 4, &c, $a^p$, not greater than $N$: and in this series it is obvious that every $a^{\text{th}}$ term is a multiple of $a$, and therefore not prime to $N$, the number of such terms as $a$, $2a$, $3a$, $4a$, &c., $a^{p-1}a$, evidently being $a^{p-1}$:

whence, of these the number which are not multiples of $a$ is

$$=a^p-a^{p-1}=(a-1)\,a^{p-1}=N\left(\frac{a-1}{a}\right).$$

Next, let $r=$&c.$=0$, or $N=a^p b^q$; then it is evident that the $a^{p-1}$ multiples of $a$ comprised in the numbers 1, 2, 3, 4, &c., $a^p$, being combined with each of the terms 1, $b$, $b^2$, &c., $b^q$, will, by (307), give $a^{p-1}b^q$ numbers less than $N$ divisible by $a$: similarly, we shall have $a^p b^{q-1}$ numbers less than $N$ divisible by $b$, and $a^{p-1}b^{q-1}$ divisible by $ab$; and it is manifest that these latter $a^{p-1}b^{q-1}$ numbers are likewise included in the two former sets: whence it follows that there are less than $N$

$a^{p-1}b^q-a^{p-1}b^{q-1}=(b-1)\,a^{p-1}b^{q-1}$ numbers divisible by $a$ only, and

$a^p b^{q-1} - a^{p-1} b^{q-1} = (a-1) a^{p-1} b^{q-1}$ numbers divisible by $b$ only:

therefore the number of numbers less than $N$ which involve neither $a$, $b$ nor $ab$, will evidently be

$$= a^p b^q - (b-1) a^{p-1} b^{q-1} - (a-1) a^{p-1} b^{q-1} - a^{p-1} b^{q-1}$$

$$= (ab - a - b + 1) a^{p-1} b^{q-1}$$

$$= (a-1)(b-1) a^{p-1} b^{q-1} = N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right);$$

and the same mode of reasoning, when $N = a^p b^q c^r$ &c., will lead to the conclusion, that the number of integers, unity included, less than $N$ and prime to it, is expressed by

$$(a-1)(b-1)(c-1) \text{ \&c. } a^{p-1} b^{q-1} c^{r-1} \text{ \&c.};$$

$$\text{or } N\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right) \text{ \&c.}$$

415. Cor. Hence, in (392), the number of forms of prime numbers to any given modulus may be determined.

For, let $4A$ be the modulus used; then it is manifest that the number of forms will be equal to the number of integers that are less than $2A$ and prime to it:

now, if $2A = a^p b^q c^r$ &c., we have just seen that the number of integers less than $2A$ and prime to it

$$= 2A\left(\frac{a-1}{a}\right)\left(\frac{b-1}{b}\right)\left(\frac{c-1}{c}\right) \text{ \&c.};$$

which, therefore, expresses the number of forms of prime numbers to the given modulus $4A$.

From this it appears, that the numbers having the least numbers of integers less than and prime to their halves, may be most advantageously employed as the moduli for forms expressive of prime numbers.

Ex. 1. Required the number of numbers less than 30 which are prime to it.

Here, $30 = 2.3.5$; therefore the number required will be

$$= 30\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right)\left(\frac{5-1}{5}\right) = 8;$$ and the numbers are

1, 7, 11, 13, 17, 19, 23, 29.

Ex. 2. Required the number of forms for prime numbers when the modulus is 20.

Here $10 = 2.5$; whence we obtain the number required

$$= 10\left(\frac{2-1}{2}\right)\left(\frac{5-1}{5}\right) = 4;$$

and the forms themselves will be

$$20m \pm 1,\ 20m \pm 3,\ 20m \pm 7 \text{ and } 20m \pm 9.$$

## IV. Formation &c. of Polygonal Numbers.

416. Def. *Polygonal Numbers* are the sums of any numbers of terms of certain arithmetical series in each of which the first term is unity; and they are distinguished into orders dependent upon the common difference.

417. If the common differences of the arithmetical series be 0, 1, 2, 3, 4, &c. we shall have, by means of the expression,

$$S = \{2a + (n-1)\,d\}\,\frac{n}{2},$$

the general terms of the corresponding orders of polygonal numbers equal to

$$n,\ \frac{n^2+n}{2},\ \frac{2n^2+0n}{2},\ \frac{3n^2-n}{2},\ \frac{4n^2-2n}{2},\ \&c.;$$

and the numbers themselves will be found by giving to $n$ the values, 1, 2, 3, &c. in succession.

418. Cor. 1. Hence, if, for the sake of uniformity of system, we designate a series of units by the name of the first

order, we shall have the following list of polygonal numbers, in the orders to which they belong:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1. | *Units,*................... | 1, | 1, | 1, | 1, | 1, | 1, | &c.; |
| 2. | *Lineal Numbers,*....... | 1, | 2, | 3, | 4, | 5, | 6, | &c.; |
| 3. | *Triagonal Numbers,*... | 1, | 3, | 6, | 10, | 15, | 21, | &c.; |
| 4. | *Quadragonal Numbers,* | 1, | 4, | 9, | 16, | 25, | 36, | &c.; |
| 5. | *Pentagonal Numbers,* . | 1, | 5, | 12, | 22, | 35, | 51, | &c.; |
| | &c........................................... | | | | | | | |

and in this arrangement, if $r$ be the denomination of the order, the common difference of the corresponding arithmetic series will be $r-2$, and we shall obviously have the $n^{\text{th}}$ or general term of the polygonal numbers of the $r^{\text{th}}$ order equal to

$$\frac{(r-2)\,n^2-(r-4)\,n}{2};$$

from which the polygonal numbers belonging to all the orders may be derived, by assigning the requisite value to $r$.

Thus, the *r-gonal* numbers are $1$, $r$, $3r-3$, $6r-8$, $10r-15$, $15r-24$, &c.

419. Cor. 2. Numbers thus formed are termed polygonal, from the circumstance of their capability of being represented by the figures whose names they bear, the sides of the polygons corresponding to the values of $n$ in the formula above given.

Thus, if a dot be taken to represent each of the units in $n$, we may arrange these dots, when the values of $n$ are 1, 2, 3, &c., in the following order:

1. *Points,*.............. . . . &c.:
2. *Lines,* .............. . .. ... &c.:
3. *Trigons,*............ . ∴ ⁖ &c.:
4. *Squares,*............ . :: ⁞⁞⁞ &c.:

&c....................................

which are perhaps fanciful representations from which their names may have been derived, rather than arrangements having any connection with the origin of the numbers themselves.

420. Cor. 3. If $N_3$ denote any triagonal or triangular number expressed generally by $\frac{n(n+1)}{2}$, we shall obviously have

$$8N_3+1=4(n^2+n)+1$$
$$=4n^2+4n+1=(2n+1)^2;$$

that is, every triagonal number multiplied by 8 and increased by 1, becomes a quadragonal or square number.

Again, if $r=6$, we have the $n^{\text{th}}$ term in the series of hexagonal numbers

$$=\frac{4n^2-2n}{2}=\frac{(2n-1)\,2n}{2},$$

which is manifestly the $(2n-1)^{\text{th}}$ term in the series of triagonal numbers: and similarly in other instances.

421. Cor. 4. If the magnitude $P_r$ of a polygonal number of the $r^{\text{th}}$ order be given, its place in that order, or what is usually termed its *Root*, may be found by the solution of the quadratic equation

$$\frac{(r-2)\,n^2-(r-4)\,n}{2}=P_r,$$

from which is obtained

$$n=\frac{r-4+\sqrt{8\,(r-2)\,P_r+(r-4)^2}}{2.(r-2)}.$$

Ex. Required the place of 51 in the series of pentagonal numbers.

In this instance, the denomination $r$ of the order being 5, we have

$$n=\frac{1+\sqrt{8\,.\,3\,.\,51+1}}{2\,.\,3}=\frac{1+\sqrt{1225}}{6}=6;$$

that is, 51 is the sixth in the order of pentagonal numbers.

422. *To find the sum of* n *terms of the* r^th^ *order of polygonal numbers.*

Since, by (418), the general term of the polygonal series

$$= \frac{(r-2)\,n^2 - (r-4)\,n}{2} = \frac{(r-2)\,(n^2-n)+2n}{2}$$

$$= n\,(n-1)\left(\frac{r-2}{2}\right) + n;$$

we shall manifestly have the sum of $n$ terms of the said series

$$= \{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \&c. + (n-1)\,n\} \left\{\frac{r-2}{2}\right\}$$

$$+ 1 + 2 + 3 + \&c. + n:$$

$$\text{but } 1 \cdot 2 = 1^2 + 1,$$

$$2 \cdot 3 = 2^2 + 2,$$

$$3 \cdot 4 = 3^2 + 3,$$

$$\&c. \ldots\ldots\ldots\ldots$$

$$(n-1)\,n = (n-1)^2 + n - 1:$$

$$\therefore\ 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \&c. + (n-1)\,n$$

$$= 1^2 + 2^2 + 3^2 + \&c. + (n-1)^2 + 1 + 2 + 3 + \&c. + (n-1)$$

$$= \frac{(n-1)\,n\,(2n-1)}{1 \cdot 2 \cdot 3} + \frac{(n-1)\,n}{2}, \text{ by (262)},$$

$$= \frac{(n-1)\,n\,(n+1)}{3}: \text{ also } 1 + 2 + 3 + \&c. + n = \frac{n\,(n+1)}{2};$$

wherefore the sum of the polygonal series becomes

$$= \frac{(n-1)\,n\,(n+1)\,(r-2)}{2 \cdot 3} + \frac{n\,(n+1)}{2}$$

$$= \frac{n\,(n+1)}{1 \cdot 2}\left\{\frac{(n-1)\,(r-2)+3}{3}\right\}.$$

Ex. Let $r$ be taken equal to 2, 3, 4, 5, 6, &c. in succession, and denoting the sums of the corresponding orders by $S_2$, $S_3$, $S_4$, $S_5$, $S_6$, &c. we obtain

$$S_2 = \frac{n(n+1)(0n+3)}{1.2.3};$$

$$S_3 = \frac{n(n+1)(1n+2)}{1.2.3};$$

$$S_4 = \frac{n(n+1)(2n+1)}{1.2.3};$$

$$S_5 = \frac{n(n+1)(3n+0)}{1.2.3};$$

$$S_6 = \frac{n(n+1)(4n-1)}{1.2.3}; \text{ \&c.}$$

423. The last article furnishes us with the means of ascertaining the number of *Balls*, *Shot* or *Shells* forming any regular *Pile*.

Whenever a pile of balls is complete, it will manifestly finish with a single ball, the number of horizontal courses being the same as the number of balls in one side of the lowest course: consequently, the numbers of balls constituting all such piles, will be represented by the sums of the series of triangular, square, &c. numbers, whose number of terms is equal to the number contained in each side of its base.

Ex. 1. Find the number of shot in a finished triangular pile, the number in one side of the base being 40.

Generally, for triangular numbers, we have

$$S_3 = \frac{n(n+1)(n+2)}{1.2.3};$$

and in this instance $n = 40$:

$$\therefore \text{ the number of shot in the pile} = \frac{40.41.42}{1.2.3}$$

$$= 11480.$$

Ex. 2. Required the number of shells contained in a square pile, whose side consists of 20.

Here, by (422), we have $S_4 = \frac{n(n+1)(2n+1)}{1 . 2 . 3}$, which when $n$ is made equal to 20, gives the required number

$$= \frac{20 . 21 . 41}{1 . 2 . 3} = 2870.$$

424. Cor. 1. To find the number of shot in a broken pile of the kind above described, we have merely to compute the numbers which would be contained in the entire pile, were it finished, and in the part which is wanting; and then to take their difference.

Ex. What number of shot is contained in five courses of an unfinished pentagonal pile, when each side of the lowest course consists of 12?

Since, by (422), $S_5 = \frac{n^2(n+1)}{1 . 2}$, we have the number in the whole pile $= \frac{12 . 12 . 13}{1 . 2} = 936$: also, the number which would be contained in the part wanting will obviously be found by making $n = 12 - 5 = 7$, and therefore $= \frac{7 . 7 . 8}{1 . 2} = 196$: therefore the number in the broken pile $= 936 - 196 = 740$.

425. Cor. 2. It is sometimes the practice to pile balls in horizontal courses forming rectangles, which consequently finish in a single row at the top: and it is manifest that the number of courses will, in such cases, be equal to the number of balls in the breadth of the lowest, whilst the number in the finishing row will exceed by 1 the difference of the numbers in the length and breadth of the base. The formulæ above referred to will not enable us to determine the numbers of balls in such piles, but we may readily deduce one which will answer that purpose.

Let $p$ and $q$ represent the numbers in the length and breadth of the lowest course, $n$ the number of courses one upon another: then will

$$p-1,\ q-1;\ p-2,\ q-2;\ \&c.,\ p-n+1,\ q-n+1,$$

be the numbers in the lengths and breadths of the second, third, &c., $n^{\text{th}}$ courses: and the entire number of shot in this pile will obviously be represented by

$$pq+(p-1)(q-1)+(p-2)(q-2)+\&c.+(p-n+1)(q-n+1)$$

$$=pq+pq-(p+q)+1^2+pq-2(p+q)+2^2+\&c.$$

$$+pq-(n-1)\ (p+q)+(n-1)^2$$

$$=npq-\{1+2+3+\&c.+(n-1)\}(p+q)$$

$$+1^2+2^2+3^2+\&c.+(n-1)^2$$

$$=npq-\frac{(n-1)\,n}{1\,.\,2}(p+q)+\frac{(n-1)\,n\,(2n-1)}{1\,.\,2\,.\,3}$$

$$=\frac{n}{4}\left\{4pq-2(n-1)\,(p+q)+\frac{2(n-1)\,(2n-1)}{3}\right\}$$

$$=\frac{n}{4}\left\{(2p-n+1)\,(2q-n+1)+\frac{(n-1)\,(n+1)}{3}\right\};$$

which enunciated at length is the common practical rule.

If $n=q$, or the pile be a finished one, we shall have the number in the uppermost row $=p-q+1$; and the total number in the pile

$$=\frac{q}{4}\left\{(2p-q+1)\,(q+1)+\frac{(q-1)\,(q+1)}{3}\right\}:$$

and this, when $q=p$, becomes

$$=\frac{p}{4}\left\{(p+1)^2+\frac{p^2-1}{3}\right\}=\frac{p\,(p+1)\,(2p+1)}{1\,.\,2\,.\,3},$$

the number in a completed square pile, as shewn before.

## V. Formation &c. of Figurate Numbers.

426. Def. *Figurate Numbers* are those, whose $n^{\text{th}}$ or general terms are comprised in the formula

$$N = \frac{n(n+1)(n+2)(n+3)\,\&c.\,(n+r)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \&c.\,(r+1)};$$

and they are distinguished into the first, second, third, &c. orders by assigning to $r$ the values 1, 2, 3, &c. respectively.

427. Cor. 1. Hence, corresponding to the values 1, 2, 3, 4, &c. of $r$, we have, by making $n$ equal to 1, 2, 3, 4, &c. in succession, the following orders of figurate numbers:

$$\textit{First order};\ 1,\ 3,\ 6,\ 10,\ \&c.,\ \frac{n(n+1)}{1 \cdot 2};$$

$$\textit{Second order};\ 1,\ 4,\ 10,\ 20,\ \&c.,\ \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3};$$

$$\textit{Third order};\ 1,\ 5,\ 15,\ 35,\ \&c.,\ \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4};$$

$$\textit{Fourth order};\ 1,\ 6,\ 21,\ 56,\ \&c.,\ \frac{n(n+1)(n+2)(n+3)(n+4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5};$$

&c.................................

and it is obvious that by similar substitutions for $n$, the $r^{\text{th}}$ order will be

$$1,\ \frac{r+2}{1},\ \frac{(r+2)(r+3)}{1 \cdot 2},\ \frac{(r+2)(r+3)(r+4)}{1 \cdot 2 \cdot 3},\ \&c.,$$

$$\frac{(r+2)(r+3)(r+4)\ \&c.\ (r+n)}{1 \cdot 2 \cdot 3 \cdot \&c.\,(n-1)}.$$

428. Cor. 2. By the formula from which they are generated, it appears that the general terms of figurate numbers of the first, second, third, &c. orders, are the co-efficients of the $n^{\text{th}}$ terms of the expansions of $(1+x)^{n+1}$, $(1+x)^{n+2}$, $(1+x)^{n+3}$, &c., respectively.

429. *If the* $(n+1)^{\text{th}}$ *term of the* $r^{\text{th}}$ *order of figurate numbers be multiplied by* n, *the product is equal to* $(r+2)$ *times the* $n^{\text{th}}$ *term of the* $(r+1)^{\text{th}}$ *order.*

For, $n$ times the $(n+1)^{\text{th}}$ term of the $r^{\text{th}}$ order

$$= n \times \frac{(n+1)(n+2)(n+3) \,\&\text{c.}\, (n+r+1)}{1\,.\,2\,.\,3\,.\,4\,.\,\&\text{c.}\,(r+1)}$$

$$= (r+2) \times \frac{n(n+1)(n+2)(n+3) \,\&\text{c.}\, (n+r+1)}{1\,.\,2\,.\,3\,.\,4\,.\,\&\text{c.}\,(r+2)}$$

$= (r+2)$ times the $n^{\text{th}}$ term of the $(r+1)^{\text{th}}$ order.

430. Cor. From this article it appears that the $n^{\text{th}}$ term of the $(r+1)^{\text{th}}$ order of figurate numbers $= \dfrac{n}{r+2}$ times the $(n+1)^{\text{th}}$ term of the $r^{\text{th}}$ order, and thus the terms of any order may be determined from those of the order which immediately precedes it.

431. *If the* $n^{\text{th}}$ *term of the* $r^{\text{th}}$ *order of figurate numbers be added to the* $(n+1)^{\text{th}}$ *term of the* $(r-1)^{\text{th}}$ *order, the sum will be the* $(n+1)^{\text{th}}$ *term of the* $r^{\text{th}}$ *order.*

For, the $n^{\text{th}}$ term of the $r^{\text{th}}$ order + the $(n+1)^{\text{th}}$ term of the $(r-1)^{\text{th}}$ order

$$= \frac{n(n+1)(n+2) \,\&\text{c.}\, (n+r)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,(r+1)} + \frac{(n+1)(n+2)(n+3) \,\&\text{c.}\, (n+r)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,r}$$

$$= \frac{(n+1)(n+2)(n+3) \,\&\text{c.}\, (n+r)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,r}\left\{\frac{n}{r+1}+1\right\}$$

$$= \frac{(n+1)(n+2)(n+3) \,\&\text{c.}\, (n+r)(n+r+1)}{1\,.\,2\,.\,3\,.\,\&\text{c.}\,r\,.\,(r+1)},$$

which is obviously the $(n+1)^{\text{th}}$ term of the $r^{\text{th}}$ order.

432. Cor. 1. By this proposition is immediately discovered the law of the formation of the terms of any order by

means of the terms of the preceding order; for, if to the $n^{\text{th}}$ term of any order there be added the $(n+1)^{\text{th}}$ term of the next inferior order, the sum will be the $(n+1)^{\text{th}}$ term of that order: and this will be found to obtain in the orders as stated in (427).

433. Cor. 2. Also, since the first term in every order is 1, it follows that the second term of any order is equal to the sum of the first two terms of the next inferior order: the third term to the first three terms of the preceding order, and generally the $n^{\text{th}}$ term of any order is equal to the sum of the first $n$ terms of the order which immediately precedes it. This will readily be observed to be true by reference to the numbers given in (427).

434. *To find the sum of* n *terms of the* r$^{\text{th}}$ *order of figurate numbers.*

By reversing the latter part of the last article, we have the sum of $n$ terms of the $r^{\text{th}}$ order equal to the $n^{\text{th}}$ term of the $(r+1)^{\text{th}}$ order $= \dfrac{n(n+1)(n+2) \text{ \&c. } (n+r+1)}{1 \,.\, 2 \,.\, 3 \,.\, \text{\&c. } (r+2)}$.

Ex. If we make $r$ equal to 1, 2, 3, &c. in succession, and denote the corresponding sums by $S_1$, $S_2$, $S_3$, &c. we obtain

$$S_1 = \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3};$$

$$S_2 = \frac{n(n+1)(n+2)(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4};$$

$$S_3 = \frac{n(n+1)(n+2)(n+3)(n+4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5};$$

&c.......................................

435. Since the figurate numbers of different orders are composed of the sums of series of polygonal numbers, they may be conceived to form pyramids in the same manner as a series of polygons each less than the other, but of the same number

of sides when applied to one another would form a pyramid, and on that account it was formerly usual to term them *Pyramidal* numbers of different orders.

436. The connection subsisting between figurate numbers and the expansions of binomials alluded to in (428), induced the earlier mathematicians to pay considerable attention to the laws of their formation, by means of which they obtained the expansion of one power from that which immediately precedes it, the sets of numbers

$$
\begin{array}{llllll}
1, & 1, & 1, & 1, & 1, & 1, \text{ \&c.;} \\
1, & 2, & 3, & 4, & 5, & 6, \text{ \&c.;} \\
1, & 3, & 6, & 10, & 15, & 21, \text{ \&c.;} \\
1, & 4, & 10, & 20, & 35, & 56, \text{ \&c.;} \\
1, & 5, & 15, & 35, & 70, & 126, \text{ \&c.;} \\
1, & 6, & 21, & 56, & 126, & 252, \text{ \&c.;} \\
\text{\&c.} \ldots\ldots & & & & &
\end{array}
$$

whether read horizontally or vertically being termed *Binomial Columns.*

Instead of the law of their generation being deduced from their general forms as was first done by LEGENDRE, the forms of figurative numbers were then determined from the consideration of the manner in which they were produced, which added greatly to the difficulty of the subject; and in this point of view it was treated by FERMAT and others; but the demonstration of the Binomial Theorem in its present shape has now rendered these numbers matters of curiosity rather than of use.

For much important information upon the subjects briefly treated of in this Chapter, the reader is referred to the *Essai sur la Theorie des Nombres par* A. M. LEGENDRE, and to BARLOW's *Elementary Investigation of the Theory of Numbers.*

# APPENDIX.

## EXAMPLES FOR PRACTICE.

### CHAP. I.

When the symbols $a$, $b$, $c$, $d$, $e$, &c. are assumed as the Algebraical representatives of the natural numbers 1, 2, 3, 4, 5, &c. it is required to prove that

1. $2a+3b-4c+7d=24$.

2. $5ab+7bd-8ac+15cd=222$.

3. $(12a-b)(25a-3c)(30a-5d)=1600$.

4. $a^2+b^2-(c+d)^2+3e^2-ef=1$.

5. $\dfrac{2d^2-e^2}{de}+\dfrac{3ad-5b}{2}=1\dfrac{7}{20}$.

6. $\sqrt{2(a^2+ab+b^2)+3(2c-d)^2-be}=4$.

7. $\sqrt[3]{10a^3+7a^2b-ab^2+3a^2c+bd^2+d^3}=5$.

8. In Algebra, the quantities concerned are denoted by the letters of the alphabet and the operations performed upon them by signs invented for the purpose.

9. When the signs + and − are placed before any quantities, which are therefore sometimes termed *additive* and *subtractive*, they may be considered to express the *qualities* or *affections* of the said quantities.

10. If $a$, $b$ and $x$ be taken to denote the numbers 21, 17 and 4, the expression, "the excess of 21 above 17 is 4", may be symbolized by $a-b=x$.

## CHAP. II.

### I. ADDITION.

1. The Algebraical Sum of $12a+5c+17d+13b$, $8a+12b+15d+8c$, $11c+15a+23b+10d$ and $4d+3a+20b+18c$ is $38a+68b+42c+46d$.

2. Of $15a+14b+13c+17d+18$, $3a+12b+17c+20d+14$, and $18a+28b+4c+24d+44$ is $36a+54b+34c+61d+76$.

3. Of $ab+4ax+3cy+2ez$, $14ax+20ez+19ab+8cy$ and $13cy+21ez+15ax+24ab$ is $44ab+33ax+24cy+43ez$.

4. Of $a+b+c$, $a+b-c$, $a+c-b$ and $b+c-a$ is $2a+2b+2c=2(a+b+c)$.

5. Of $5a+3b-4c$, $2a-5b+6c+2d$, $a-4b-2c+3e$ and $7a+4b-3c-6e$ is $15a-2b-3c+2d-3e$.

6. Of $3a^2+2ab+4b^2$, $5a^2-8ab+6b^2$, $-4a^2+5ab-b^2$, $18a^2-20ab-19b^2$ and $14a^2-3ab+20b^2$ is $36a^2-24ab+10b^2$.

7. Of $4x^3-5ax^2+6a^2x-5a^3$, $3x^3+4ax^2+2a^2x+6a^3$, $-17x^3+19ax^2-15a^2x+8a^3$, $13ax^2-27a^2x+18a^3$ and $12x^3+3a^2x-20a^3$ is $2x^3+31ax^2-31a^2x+7a^3$.

8. Of $5xy-7ez+18ax-14by$, $3xy-5cd+11eg+14ez$, $13ax+20eg-35cd+18$ and $25xy-15eg+9by-12ax$ is $33xy+7ez+19ax-5by-40cd+16eg+18$.

9. Of $10a^2b-12a^3bc-15b^2c^4+10$, $-4a^2b+8a^3bc-10b^2c^4-4$, $-3a^2b-3a^3bc+20b^2c^4-3$ and $2a^2b+12a^3bc+5b^2c^4+2$ is $5a^2b+5a^3bc+5=5(a^2b+a^3bc+1)$.

10. Of $13(a+c)x^2-14y^2+20$, $15y^2-20ax+16+5xy$, $10+4(a-c)x^2+4y^2-10ax$ and $-17ax^2-9cx^2-18xy+10y^2$ is $15y^2-30ax-13xy+46$.

11. Of $ax^m + bx^n - cx^p + dx^q$, $-bx^m + ax^n - dx^p - ex^q$, $(e+f)x^m + (b+c)x^p + x^q$ and $(c-a)x^m + kx^p - (d+e)x^q$ is $(c+e+f-b)x^m + (a+b)x^n + (b-d+k)x^p + (1-2e)x^q$.

12. Of $(2a+b)x^2 + 4(3c+d)xy - 7(ac+bd)y$, $(4a-3b)x^2 + 4(3d-c)xy - 4(ac+3bd)y$, $(a+4b)x^2 + 4(2d-4c)xy - 8(3ac+4bd)y$ and $(3a-2b)x^2 + 4(4d-3c)xy - 8(4ac+5bd)y$ is $10ax^2 + 4(10d-5c)xy - (67ac+91bd)y$.

13. Of $(2a-3b)ax + (5a-3c)xy - (4a+5)by$, $(5a-2b+c)ax - (8a+3b-2c)by + (3a-2b)xy$, $(2a-5)xy + (4b-3a+7c)ax - (6b-4a+20)by$ and $-(4c-3a-9b)by + (b-4a-8c)ax + (a+3c)xy$ is $(11a-2b-5)xy - (5a-2c-25)by$.

## II. SUBTRACTION.

1. The Algebraical Excess of $6a^2+12ab+19b^2+c^2$ above $4a^2+8ab+13b^2$ is $2a^2+4ab+6b^2+c^2$.

2. Of $11a^2+12ab+4b^2+7ac+9c^2$ above $7a^2+19ab+5b^2+13ac+2c^2$ is $4a^2-7ab-b^2-6ac+7c^2$.

3. Of $5a^2+4ab-3ac+bc-3c^2$ above $3a^2+3ab+3bc-2c^2$ is $2a^2-3ac+ab-2bc-c^2$.

4. Of $12x+6a-4b-12c-7e-5f$ above $2x-3a+4b-5c+6d-7e$ is $10x+9a-8b-7c-6d-5f$.

5. Of $28ax^3-16a^2x^2+25a^3x-13a^4$ above $18ax^3+20a^2x^2-24a^3x-7a^4$ is $10ax^3-36a^2x^2+49a^3x-6a^4$.

6. Of $8a^2xy-5bx^2y+17cxy^2-9y^5$ above $a^2xy+3bx^2y-13cxy^2+20y^5$ is $7a^2xy-8bx^2y+30cxy^2-29y^5$.

7. Of $2ax^4-(b+ac)x^3+(a+c)x^2-(b+d)x$ above $ax^4-acx^3+ax^2-bx$ is $ax^4-bx^3+cx^2-dx$.

8. Of $ax^3-bx^2+cx-d$ above $x^3-px^2+qx-r$ is $(a-1)x^3-(b-p)x^2+(c-q)x-(d-r)$.

9. Of $129a^3 - 11a(a^2+b^2) + 10(b+c)ax - 8b^3$ above $54a^3 - 27a(a^2+b^2) + 14(b+c)ax - 13b^3$ is $75a^3 + 16a(a^2+b^2) - 4(b+c)ax + 5b^3$.

10. Of $(a^2+bc)x^2 - (a^2-c^2)bx + (ae+fg)xy + 7(a^4-5b^4)$ above $bcx^2 - (a^2-b^2)bx - (cd-fg)xy - 35b^4$ is $a^2x^2 - (b^2-c^2)bx + (ae+cd)xy + 7a^4$.

11. Of $(2x^2+3ay)y^2 + (125c^2-d^2)xy - 4cd(y^2-x^2) - 25cd$ above $(x^2-ay)y^2 - 25(a^2-5c^2)xy + 13cd(x^2-y^2) + 50ce$ is $(x^2+4ay)y^2 + (25a^2-d^2)xy - 9cd(x^2-y^2) - 25c(d+2e)$.

12. Of $2x(x^2+ax+a^2) - 3y(y^2-by+b^2) + 4z(z^3-c^3)$ above $x(x^2-ax+a^2) + 2y(y^2+by+b^2) - 3z(c^3-z^3)$ is $x(x^2+3ax+a^2) - y(5y^2-by+5b^2) + z(z^3-c^3)$.

## III. MULTIPLICATION.

1. The Algebraical Product of $ax^2y^3$ by $bxy$ is $abx^3y^4$.

2. Of $mx^2$ by $-nxy^3$ is $-mnx^3y^3$ and of $-acx$ by $-2ayz$ is $2a^2cxyz$.

3. Of $a^2+ax+x^2$ by $x^2y^2$ is $a^2x^2y^2 + ax^3y^2 + x^4y^2$ and of $x^2-xy+y^2$ by $-a^2b^2$ is $-a^2b^2x^2 + a^2b^2xy - a^2b^2y^2$.

4. Of $3x+2y$ by $2x+3y$ is $6x^2+13xy+6y^2$.

5. Of $3ab+4b^2$ by $2ab-3b^2$ is $6a^2b^2 - ab^3 - 12b^4$.

6. Of $27x^3+9x^2y+3xy^2+y^3$ by $3x-y$ is $81x^4-y^4$.

7. Of $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ by $a+2b$ is $a^5+32b^5$.

8. Of $x^{15}-x^{12}y^2+x^9y^4-x^6y^6+x^3y^8-y^{10}$ by $x^3+y^2$ is $x^{18}-y^{12}$.

9. Of $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$ by $x^2 - 2y^2$ is $x^6 - 2x^5y + 2x^4y^2 - 4x^3y^3 + 8x^2y^4 + 16xy^5 - 32y^6$.

10. Of $x^3 \mp x^2y + xy^2 \mp y^3$ by $x \pm y$ is $x^4 - y^4$.

11. Of $x^2 + 2xy + 3y^2$ by $x^2 - 2xy + y^2$ is $x^4 - 4xy^3 + 3y^4$.

12. Of $a^3 + 2a^2b + 3ab^2 + 4b^3$ by $a^2 - 2ab - 3b^2$ is $a^5 - 4a^3b^2 - 8a^2b^3 - 17ab^4 - 12b^5$.

13. Of $14a^3c - 6a^2bc + c^2$ by $14a^3c + 6a^2bc - c^2$ is $196a^6c^2 - 36a^4b^2c^2 + 12a^2bc^3 - c^4$.

14. Of $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 - 2x + 1$ is $x^6 - 2x^3 + 1$.

15. Of $-b^3 + ab^2 - a^2b + a^3$ by $-4b^2 - 3ab + 3a^2$ is $4b^5 - ab^4 - 2a^2b^3 + 2a^3b^2 - 6a^4b + 3a^5$.

16. Of $2a^4 - 3a^3b - 5a^2b^2$ by $a^3 - 2a^2b + 3ab^2$ is $2a^7 - 7a^6b + 7a^5b^2 + a^4b^3 - 15a^3b^4$.

17. Of $x^3 + 2ax^2 + 2a^2x + a^3$ by $x^3 - 2ax^2 + 2a^2x - a^3$ is $x^6 - a^6$.

18. Of $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$ by $a^3 - a^2b + ab^2 - b^3$ is $a^8 + a^6b^2 - a^2b^6 - b^8$.

19. Of $x^2 + ax + b$ by $x^2 - ax + c$ is $x^4 + (b + c - a^2)x^2 - (b - c)ax + bc$.

20. Of $a^3b + a^4 + a^3b^2 - a^4b + a^4b^2$ by $-a^3 - a^2b^2 - a^3b + a^2b$ is $(1 - b^2)b^2a^5 - (4 - b + b^2)b^2a^6 - (1 + b^3)a^7$.

21. Of $(b - c)a^3 + (b^3 - b^2c + bc^2 - c^3)a^2 + 3a$ by $b + c$ is $(b^2 - c^2)a^3 + (b^4 - c^4)a^2 + 3(b + c)a$.

22. Of $a^{m-1} \pm b^{m-1}$ by $a^{n+1} \pm b^{n+1}$ is $a^{m+n} \pm a^{n+1}b^{m-1} \pm a^{m-1}b^{n+1} + b^{m+n}$.

23. Of $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$ is $a^3 + b^3 + c^3 - 3abc$.

24. Of $x^2+(a+b)x+(a^2+ab+c)$ by $x-a$ is $x^3+bx^2+cx-a^3-a^2b-ac$.

25. Of $a-(a+b)x+(a+b+c)x^2-(a+b+c+d)x^3+(a+b+c+d+e)x^4$ by $1+x$ is $a-bx+cx^2-dx^3+ex^4+(a+b+c+d+e)x^5$.

26. The continued product of $1+x$, $1+x^4$ and $1-x+x^2-x^3$ is $1-x^8$.

27. Of $a^2+ab+b^2$, $a^3-a^2b+b^3$ and $a-b$ is $a^6-a^5b+a^2b^4-b^6$.

28. Of $x+a$, $x-b$ and $x+c$ is $x^3+(a-b+c)x^2-(ab-ac+bc)x-abc$.

29. Of $x-4$, $x+4$, $x+3$ and $x-3$ is $x^4-25x^2+144$.

30. Of $a+b$, $a-b$, $a^2+ab+b^2$ and $a^2-ab+b^2$ is $a^6-b^6$.

31. Of $a+b+c$, $a+b-c$, $a+c-b$ and $b+c-a$ is $2(a^2b^2+a^2c^2+b^2c^2)-a^4-b^4-c^4$.

32. Of $a+b+c-d$, $a+b+d-c$, $a+c+d-b$ and $b+c+d-a$ is $8abcd+2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2)-a^4-b^4-c^4-d^4$.

33. Of $x+a$, $x+b$, $x+c$ and $x+d$ is $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x+abcd$.

34. Of $x+4$, $x+10$, $x-7$, $x-9$ and $x+2$ is $x^5-125x^3+3004x+5040$.

35. Of $x+8$, $x-8$, $x+5$, $x-5$, $x+3$ and $x-3$ is $x^6-98x^4+2401x^2-14400$.

## IV. DIVISION.

1. The Algebraical Quotient of $2a^2b-6a^2c+4abc$ by $2a$ is $ab-3ac+2bc$.

2. Of $5x^3y^3-40a^2x^2y^2+25a^4xy$ by $-5xy$ is $-x^2y^2+8a^2xy-5a^4$.

3. Of $x^m-ax^{m-1}+bx^{m-2}-cx^n$ by $x^n$ is $x^{m-n}-ax^{m-n-1}+bx^{m-n-2}-c$.

4. Of $8a^2-26ab+15b^2$ by $4a-3b$ is $2a-5b$.

5. Of $x^3+6x^2+9x+4$ by $x+4$ is $x^2+2x+1$.

6. Of $a^4-a^2b^2-12b^4$ by $a^2+3b^2$ is $a^2-4b^2$.

7. Of $x^4-81y^4$ by $x-3y$ is $x^3+3x^2y+9xy^2+27y^3$.

8. Of $x^3-3x-2$ by $x^2-2x+1$ is $x+2$.

9. Of $x^3-5x^2-x+14$ by $x^2-3x-7$ is $x-2$.

10. Of $6x^3-16x^2y+6xy^2+4y^3$ by $3x^2-2xy-y^2$ is $2x-4y$.

11. Of $x^4-9x^2-6xy-y^2$ by $x^2+3x+y$ is $x^2-3x-y$.

12. Of $x^4-6x^3y+9x^2y^2-4y^4$ by $x^2-3xy+2y^2$ is $x^2-3xy-2y^2$.

13. Of $12a^4-26a^3b-8a^2b^2+10ab^3-8b^4$ by $3a^2-2ab+b^2$ is $4a^2-6ab-8b^2$.

14. Of $x^5\pm y^5$ by $x\pm y$ is $x^4\mp x^3y+x^2y^2\mp xy^3+y^4$.

15. Of $a^{m+n}-a^mb^n+a^nb^m-b^{m+n}$ by $a^m+b^m$ is $a^n-b^n$.

16. Of $a^5+a^3b^2+a^2b^3+b^5$ by $a^2-ab+b^2$ is $a^3+a^2b+ab^2+b^3$.

17. Of $a^8+a^6b^2+a^4b^4+a^2b^6+b^8$ by $a^4+a^3b+a^2b^2+ab^3+b^4$ is $a^4-a^3b+a^2b^2-ab^3+b^4$.

18. Of $ax^3 - a^2x + x^m - a^2x^{m-2}$ by $x - a$ is $ax^2 + a^2x + x^{m-1} + ax^{m-2}$.

19. Of $3a^2 + 8ab + 4b^2 + 10ac + 8bc + 3c^2$ by $a + 2b + 3c$ is $3a + 2b + c$.

20. Of $x^3 + y^3 + z^3 - 3xyz$ by $x^2 + y^2 + z^2 - xy - xz - yz$ is $x + y + z$.

21. Of $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$ is $a^3 + 2a^2b + 2ab^2 + b^3$.

22. Of $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ by $1 - 3x + 3x^2 - x^3$ is $1 - 2x + x^2$.

23. Of $x^3 - nax^2 + na^2x - a^3$ by $x - a$ is $x^2 - (n-1)ax + a^2$.

24. Of $a^2 + (a-1)x^2 + (a-1)x^3 + (a-1)x^4 - x^5$ by $a - x$ is $a + x + x^2 + x^3 + x^4$.

25. Of $1 - 9x^8 - 8x^9$ by $1 + 2x + x^2$ is $1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - 8x^7$.

26. Of $1 + x - 17x^8 + 15x^9$ by $1 - 2x + x^2$ is $1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + 13x^6 + 15x^7$.

27. Of $x^3 - 2ax^2 + (a^2 - ab - b^2)x + a^2b + ab^2$ by $x^2 - (a-b)x - ab$ is $x - a - b$.

28. Of $a + (a+b)x + (a+b+c)x^2 + (a+b+c)x^3 + (b+c)x^4 + cx^5$ by $a + bx + cx^2$ is $1 + x + x^2 + x^3$.

29. Of $x^5 - bx^4 + cx^3 - cx^2 + bx - 1$ by $x - 1$ is $x^4 - (b-1)x^3 - (b-c-1)x^2 - (b-1)x + 1$.

30. Of $a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)$ by $a^2 + 2ab + b^2 - c^2$ is $a^2 - 2ab + b^2 - c^2$.

31. Of $a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd$ by $a^2 + b^2 - c^2 - d^2 + 2(ab + cd)$ is $a^2 + b^2 - c^2 - d^2 - 2(ab + cd)$.

32. Of $a^6+a^4b^2-2a^4c^2-a^2b^4+a^2c^4-b^6-2b^4c^2-b^2c^4$ by $a^2-b^2-c^2$ is $a^4+2a^2b^2+b^4-a^2c^2+b^2c^2$.

33. Of $3abx^5-(2a^2+6b^2)x^4+(a^2b+9ab^2+4ab)x^3-(6a^2b+2ab^2+3bc)x^2+(3a^2b^2+2ac)x-abc$ by $3bx^2-2ax+ab$ is $ax^3-2bx^2+3abx-c$.

34. Of
$$\begin{array}{l|l|l|l|l|l} 9a & x^5+9a & x^4+8a & x^3+20ab & x^2+6ab & x+2a \\ -6b & -17b & +32ab & -10a^2 & -4a^2 & -1 \\ & +12ab & -16b^2 & -15b^2 & +4a & \\ & -9a^2 & -12a^2 & +8a & -8b & \\ & -4b^2 & -5b & -4b & & \\ & & -3 & -1 & & \end{array}$$
by
$$\begin{array}{l|l|l|} 3a & x^2+2a & x+2a \\ -2b & -5b & -1 \end{array} \text{ is } \begin{array}{l|l|l} 3x^3-3a & x^2-2a & x+1. \\ +2b & +3b & \\ +1 & & \end{array}$$

35. Of $x^4-px^3+qx^2-rx+s$ by $x-a$ is $x^3+(a-p)x^2+(a^2-pa+q)x+a^3-pa^2+qa-r$, with a remainder $a^4-pa^3+qa^2-ra+s$.

36. Of 1 by $1-2x+x^2$ is $1+2x+3x^2+4x^3+$ &c. *in infinitum.*

37. Of $1-x$ by $1+x-x^2$ is $1-2x+3x^2-5x^3+$ &c. *in infinitum.*

## V. INVOLUTION.

1. The square, cube, and fourth power of $4ay$ are $16a^2y^2$, $64a^3y^3$ and $256a^4y^4$.

2. Of $-5ab^2x^3y^4$ the square is $25a^2b^4x^6y^8$ and the cube is $-125a^3b^6x^9y^{12}$.

3. The $m^{\text{th}}$ power of $xy^pz^q$ is $x^my^{mp}z^{mq}$ and of $-ax^ry^t$ is $\pm a^mx^{mr}y^{mt}$.

4. The square of $a+2x$ is $a^2+4ax+4x^2$ and of $x^2-by$ is $x^4-2bx^2y+b^2y^2$.

5. Of $ax+b^2x^2$ is $a^2x^2+2ab^2x^3+b^4x^4$ and of $x^2-2x+4$ is $x^4-4x^3+12x^2-16x+16$.

6. Of $x^2-x+1$ is $x^4-2x^3+3x^2-2x+1$ and of $x^3-x^2+1$ is $x^6-2x^5+x^4+2x^3-2x^2+1$.

7. Of $3a-b+5c+d$ is $9a^2-6ab+30ac+6ad+b^2-10bc-2bd+25c^2+10cd+d^2$.

8. Of $a+bx+cx^2+dx^3=a^2+2abx+(2ac+b^2)x^2+2(ad+bc)x^3+(2bd+c^2)x^4+2cdx^5+d^2x^6$.

9. Of $(a+b)x-(a-b)y$ is $(a+b)^2x^2-2(a^2-b^2)xy+(a-b)^2y^2$ or $(a^2+2ab+b^2)x^2-2(a^2-b^2)xy+(a^2-2ab+b^2)y^2$.

10. The cube of $x^2+y^2$ is $x^6+3x^4y^2+3x^2y^4+y^6$ or $x^6+y^6+3x^2y^2(x^2+y^2)$.

11. Of $x^2-ax+a^2$ is $x^6-3ax^5+6a^2x^4-7a^3x^3+6a^4x^2-3a^5x+a^6$.

12. Of $2x^2+4ax-3a^2$ is $8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6$.

13. Of $1+x+x^2+x^3$ is $1+3x+6x^2+10x^3+12x^4+12x^5+10x^6+6x^7+3x^8+x^9$.

14. The fourth power of $1\pm x$ is $1\pm 4x+6x^2\pm 4x^3+x^4$.

15. Of $a+b-c$ is $a^4+b^4+c^4+4(a^3b-a^3c+ab^3-ac^3-b^3c-bc^3)+6(a^2b^2+a^2c^2+b^2c^2)-12(a^2bc+ab^2c-abc^2)$.

16. The fifth power of $ax-by$ is $(ax-by)^3\times(ax-by)^2=(a^3x^3-3a^2bx^2y+3ab^2xy^2-b^3y^3)\times(a^2x^2-2abxy+b^2y^2)=a^5x^5-5a^4bx^4y+10a^3b^2x^3y^2-10a^2b^3x^2y^3+5ab^4xy^4-b^5y^5$.

## VI. EVOLUTION.

1. The square root of $x^2y^4z^6$ is $\pm xy^2z^3$ and of $4x^{2m}y^{4n}z^{6p}$ is $\pm 2x^my^{2n}z^{3p}$.

2. The cube root of $a^3b^6$ is $ab^2$ and of $-8a^3b^6x^9$ is $-2ab^2x^3$.

3. The fourth root of $16a^4x^4$ is $\pm 2ax$ and of $81x^4y^{12}$ is $\pm 3xy^3$.

4. The fifth root of $32a^5x^{10}y^{15}$ is $2ax^2y^3$ and the sixth root of $729x^6y^{18}z^{24}$ is $\pm 3xy^3z^4$.

5. The square root of $x^4+2a^2x^2+a^4$ is $x^2+a^2$ and of $a^2x^2-2abxy^2+b^2y^4$ is $ax-by^2$.

6. Of $x^4-2x^3+3x^2-2x+1$ is $x^2-x+1$ or $-x^2+x-1$.

7. Of $4x^6-12x^5y+29x^4y^2-30x^3y^3+25x^2y^4$ is $2x^3-3x^2y+5xy^2$.

8. Of $4a^4-12a^3b+25a^2b^2-24ab^3+16b^4$ is $2a^2-3ab+4b^2$.

9. Of $9x^2y^4-12x^3y^3+34x^4y^2-20x^5y+25x^6$ is $3xy^2-2x^2y+5x^3$.

10. Of $9-24x-68x^2+112x^3+196x^4$ is $3-4x-14x^2$ or $14x^2+4x-3$.

11. Of $4x^2y^4-12x^3y^3+17x^4y^2-12x^5y+4x^6$ is $2xy^2-3x^2y+2x^3$.

12. Of $x^2-2ax+a^2+2xy-2ay+y^2$ is $x-a+y$ or $a-x-y$.

13. Of $a^4-4a^3b+8ab^3+4b^4$ is $a^2-2ab-2b^2$ or $2b^2+2ab-a^2$.

14. Of $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$ is $1-x+x^2-x^3$ or $x^3-x^2+x-1$.

15. Of $9x^6-12x^5+10x^4-10x^3+5x^2-2x+1$ is $3x^3-2x^2+x-1$ or $1-x+2x^2-3x^3$.

16. Of $16(a^2+1)-24a(a^2+1)+41a^2$ is $4a^2-3a+4$.

17. Of $25a^6-30a^5x+9a^4x^2+10a^3x^2-6a^2x^3+x^4$ is $5a^3-3a^2x+x^2$.

18. Of $36x^4 - 12(a-2b)x^3 + (a^2 - 4ab + 4b^2)x^2$ is $6x^2 - (a-2b)x$ or $(a-2b)x - 6x^2$.

19. Of $a^{2m}x^{2n+2} + 10ca^{2m-2}x^{2n+3} - 6a^{m+1}x^{n+1} + 25c^2a^{2m-4}x^{2n+4} - 30ca^{m-1}x^{n+2} + 9a^2$ is $a^m x^{n+1} + 5ca^{m-2}x^{n+2} - 3a$.

20. Of $a^4 + b^4 + c^4 + 2(a^2b^2 - a^2c^2 - b^2c^2)$ is $a^2 + b^2 - c^2$ or $c^2 - (a^2 + b^2)$.

21. Of $a^6 + b^6 + c^6 + d^6 - 2(a^3b^3 - a^3c^3 + a^3d^3 + b^3c^3 - b^3d^3 + c^3d^3)$ is $a^3 - b^3 + c^3 - d^3$.

22. Of $(x^2 - 2ax + a^2)(x^2 + 2ax + a^2)$ is $(x-a)(x+a)$ or $x^2 - a^2$.

23. The cube root of $x^3 \pm 9x^2 + 27x \pm 27$ is $x \pm 3$.

24. Of $a^6 + 6a^5 - 40a^3 + 96a - 64$ is $a^2 + 2a - 4$.

25. Of $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6$ is $2x^2 + 4cx - 3c^2$.

26. Of $a^3 + b^3 - c^3 + 3(a^2b - a^2c + ab^2 + ac^2 - b^2c + bc^2) - 6abc$ is $a + b - c$.

27. Of $(a+b)^{6m}x^6 + 6ca^p(a+b)^{4m}x^4 + 12c^2a^{2p}(a+b)^{2m}x^2 + 8c^3a^{3p}$ is $(a+b)^{2m}x^2 + 2ca^p$.

28. Of $x^3y^3(a^3 + 3a^2 + 3a + 1)(a^3 - 3a^2 + 3a - 1) = xy(a+1)(a-1) = a^2xy - xy$.

29. The fourth root of $a^{4m} \pm 4a^{3m+n} + 6a^{2m+2n} \pm 4a^{m+3n} + a^{4n}$ is $a^m \pm a^n$ or $a^m(1 \pm a^{n-m})$ or $a^n(a^{m-n} \pm 1)$.

30. The fifth root of $a^5x^5 - 5a^4x^4y + 10a^3x^3y^2 - 10a^2x^2y^3 + 5axy^4 - y^5$ is $ax - y$.

31. The sixth root of $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$ is $x - 2$ or $2 - x$.

## *Miscellaneous Theorems and Problems.*

1. Prove that $(a+b)^2c^2+(a-b)^2c^2=2(a^2+b^2)c^2$.

2. $(a^2+d^2)bc+(b^2+c^2)ad=(ac+bd)(ab+cd)$.

3. $2(x^2+y^2+z^2)=(x+y)^2+(x-y)^2+2z^2$ and $2(x^2+y^2+2z^2)=(x+y)^2+(x-y)^2+(2z)^2$.

4. $a^2-2x^2=2(a\pm x)^2-(a\pm 2x)^2$ and $a^2-5x^2=5(a\pm 2x)^2-(2a\pm 5x)^2$.

5. $(a^2+cb^2)(x^2+cy^2)=(ax\pm bcy)^2+c(ay\mp bx)^2$.

6. $(a^2+b^2+c^2)(x^2+y^2)=(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2$.

7. $(a^2+b^2+c^2+d^2)(x^2+y^2)=(ax+by)^2+(cx+dy)^2+(ay-bx)^2+(cy-dx)^2$.

8. $(a^2+b^2)(c^2+d^2)(x^2+y^2)=\{(ac+bd)x+(ad-bc)y\}^2+\{(ac+bd)y-(ad-bc)x\}^2$.

9. $x^2-(a^2+1)y^2=(a^2+1)(x\pm ay)^2-\{ax\pm(a^2+1)y\}^2$ and $(a^2+1)x^2-y^2=\{(a^2+1)x\pm ay\}^2-(a^2+1)(ax\pm y)^2$.

10. $(a+3b)^2-2(a+2b)^2+(a+b)^2=2b^2$ and $(a+4b)^3-3(a+3b)^3+3(a+2b)^3-(a+b)^3=6b^3$.

11. $2a(a-b-c+d)+b^2+c^2-a^2-d^2=(a-b)^2+(a-c)^2-(a-d)^2$.

12. $(a+b+c)^2-(a^2+b^2+c^2)=a(b+c)+b(a+c)+c(a+b)$.

13. $(a+b+c)^2=(a+b)^2+(a+c)^2+(b+c)^2-(a^2+b^2+c^2)$.

14. $(a+b+c)^2+(a+b-c)^2+(a+c-b)^2+(b+c-a)^2=4(a^2+b^2+c^2)$.

15. $(a+b+c)^3+(a-b-c)^3+(b-a-c)^3+(c-a-b)^3=24abc$.

16. $(a+b+c)^3 = (a+b)^3 + (a+c)^3 + (b+c)^3 - (a^3+b^3+c^3) + 6abc$.

17. $(a+b+c)^3 - (a^3+b^3+c^3) = 3(a+b+c)(ab+ac+bc) - 3abc$.

18. $(a+b+c+d)^2 = (a+b)^2 + (a+c)^2 + (a+d)^2 + (b+c)^2 + (b+d)^2 + (c+d)^2 - 2(a^2+b^2+c^2+d^2)$.

19. $2(a+b+c+d)^2 + a^2 + b^2 + c^2 + d^2 = (a+b+c)^2 + (a+b+d)^2 + (a+c+d)^2 + (b+c+d)^2$.

20. $(a+b+c-d)^2 + (a+b+d-c)^2 + (a+c+d-b)^2 + (b+c+d-a)^2 = 4(a^2+b^2+c^2+d^2)$.

21. $(a+b+c+d)^2 = a^2 + (2a+b)b + \{2(a+b)+c\}c + \{2(a+b+c)+d\}d$.

22. $(a+b+c+d)^3 = a^3 + (3a^2+3ab+b^2)b + \{3(a+b)^2 + 3(a+b)c + c^2\}c + \{3(a+b+c)^2 + 3(a+b+c)d + d^2\}d$.

23. $(a-b+c)\times(a+b-c) = a^2 - (b-c)^2 = a^2 - b^2 + 2bc - c^2$ and $(a-b+c-d)\times(a+b-c-d) = (a-d)^2 - (b-c)^2 = a^2 - b^2 - c^2 + d^2 - 2ad + 2bc$.

24. $4a^2b^2 - (a^2+b^2-c^2)^2 = (a+b+c)(a+b-c)(a+c-b)(b+c-a)$.

25. $4(ad+bc)^2 - (a^2-b^2-c^2+d^2) = (a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)$.

26. $(ax+by+cz)^2 + (ay-bx)^2 + (az-cx)^2 + (bz-cy)^2 = (a^2+b^2+c^2)(x^2+y^2+z^2)$.

27. $(au+bx+cy+dz)^2 + (ax-bu+cz-dy)^2 + (ay-bz-cu+dx)^2 + (az+by-cx-du)^2 = (a^2+b^2+c^2+d^2)(u^2+x^2+y^2+z^2)$.

28. $(au-bt)(ay-bx) + (bv-cu)(bz-cy) + (ct-av)(cx-az) = (a^2+b^2+c^2)(tx+uy+vz) - (at+bu+cv)(ax+by+cz)$.

29. $(ax+by+cz+\&c.)^2=(a+b+c+\&c.)(ax^2+by^2+cz^2+\&c.)-ab(x-y)^2-ac(x-z)^2-bc(y-z)^2-\&c.$

30. $x^{16}-1=(x^2-1)(x^2+1)(x^4+1)(x^8+1)$ and $x^{32}-1=(x^2-1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)$.

31. $x^m-1\div x-1=x^{m-1}+x^{m-2}+\&c.+x+1$, the number of terms being $m$.

32. $x^m+1$ is divisible by $x+1$ when $m$ is odd, and $x^m-1$ is divisible by $x+1$ when $m$ is even.

33. $x^{mnp}-1$ is divisible by each of the quantities $x^m-1$, $x^n-1$ and $x^p-1$: find the last term and number of terms in each case.

34. $a^{mn}-b^{mn}$ is divisible by $a^m-b^m$: find the first and second terms, and the last and last but one, and the number of terms.

35. If $a$ be greater than $b$, prove that $a^m-b^m$ is less than $ma^{m-1}(a-b)$ and greater than $mb^{m-1}(a-b)$.

36. Shew that $x^2+y^2$ can never be less than $2xy$.

37. Prove that $x^3\pm y^3$ is never less than $x^2y\pm xy^2$.

38. Shew that $abc$ is greater than $(a+b-c)(a+c-b)(b+c-a)$, unless all the quantities are equal.

39. Whether is $a^6+a^4b^2+a^2b^4+b^6$ greater or less than $(a^3+b^3)^2$?

40. If $x^2=a^2+b^2$ and $y^2=c^2+d^2$, shew whether $xy$ is greater or less than $ac+bd$ and $ad+bc$.

41. Shew that the quotient of $a^mb-ab^m-a^mc+ac^m+b^mc-bc^m$ by $(a-b)(a-c)$ is $a^{m-2}(b-c)+a^{m-3}(b^2-c^2)+\&c.+a(b^{m-2}-c^{m-2})+(b^{m-1}-c^{m-1})$.

42. If unity be divided into any two parts, the sums formed by adding each part to the square of the other are equal.

43. If the number 2 be divided into any two parts, the difference of their squares is always equal to twice the difference of the parts themselves.

## CHAP. III.

### I. COMMON MEASURES.

1. THE greatest common measure of $ax^2 - a^2x$ and $a^2x^2 + abx$ is $ax$.

2. Of $a^3x^2 - 2a^2x^3 + ax^4$ and $a^4x^2 + a^3x^3 + a^2x^4$ is $ax^2$.

3. Of $14x^2 - 7xy$ and $10ax - 5ay$ is $2x - y$.

4. Of $ac + bd + ad + bc$ and $af + 2bx + 2ax + bf$ is $a + b$.

5. Of $4ax + 5bx - 6x^2$ and $12ayz + 15byz - 18xyz$ is $4a + 5b - 6x$.

6. Of $x^3 - 5x^2 + 7x - 3$ and $x^2 + x - 12$ is $x - 3$.

7. Of $x^3 - 3x + 2$ and $x^3 + 4x^2 - 5$ is $x - 1$.

8. Of $x^3 + 1$ and $x^3 + mx^2 + mx + 1$ is $x + 1$.

9. Of $x^4 - 1$ and $x^3 + 3x^2 - 4$ is $x - 1$.

10. Of $x^3 - 8x^2 + 21x - 18$ and $3x^3 - 16x^2 + 21x$ is $x - 3$.

11. Of $7x^2 - 12x + 5$ and $2x^3 + x^2 - 8x + 5$ is $x - 1$.

12. Of $x^3 - 3x^2 - 10x + 24$ and $2ax^3 - 10ax^2 + 8ax$ is $x - 4$.

13. Of $x^3 - 4x^2 + 9x - 10$ and $x^3 + 2x^2 - 3x + 20$ is $x^2 - 2x + 5$.

14. Of $x^3-3x^2+3x-2$ and $x^3-4x^2+6x-4$ is $x-2$.

15. Of $x^3-3x^2+7x-21$ and $2x^4+19x^2+35$ is $x^2+7$.

16. Of $a^3-4a^2b+4ab^2-b^3$ and $2a^2-6ab+4b^2$ is $a-b$.

17. Of $2x^3-8x^2y+16xy^2-16y^3$ and $8x^2-4xy-24y^2$ is $2x-4y$.

18. Of $x^3-3a^2x-2a^3$ and $x^4-ax^3+a^3x-10a^4$ is $x-2a$.

19. Of $4a^2-5ab+b^2$ and $3a^3-3a^2b+ab^2-b^3$ is $a-b$.

20. Of $x^3-19x+30$ and $x^3-2x^2-7x+14$ is $x-2$.

21. Of $2a^3+3a^2x-9ax^2$ and $6a^3x-17a^2x^2+14ax^3-3x^4$ is $2a-3x$.

22. Of $9x^2-3xy-6x+2y$ and $6x^4-4x^3-3xy^2+2y^2$ is $3x-2$.

23. Of $x^4-4x^3+8x^2-16x+16$ and $x^4-6x^3+13x^2-12x+4$ is $x^2-4x+4$.

24. Of $45a^3b+3a^2b^2-9ab^3+6b^4$ and $54a^2b-24b^3$ is $9ab+6b^2$.

25. Of $48x^3+8x^2+31x+15$ and $24x^3+22x^2+17x+5$ is $12x+5$.

26. Of $x^6+x^2y-x^4y^2-y^3$ and $x^4-x^2y-x^2y^2+y^3$ is $x^2-y^2$.

27. Of $x^8+a^2x^6+ax^2+a^3$ and $x^6-a^4x^2-ax^4+a^5$ is $x^2+a^2$.

28. Of $3x^3-(3c+d+3)x^2+(3c+d)x$ and $2x^2-(2a+b+2)x+2a+b$ is $x-1$.

29. Of $a^3+(a+1)ay+y^2$ and $a^4-a^2(y^2-y)-y^3$ is $a^3+a^2y+ay+y^2$.

30. Of $x^4 - x^3 + x - 1$ and $x^4 - 2x^3 + 3x^2 - 2x + 1$ is $x^2 - x + 1$.

31. Of $a^4 - 4ab^3 + 3b^4$ and $a^4 - a^3b - ab^3 + b^4$ is $(a-b)^2$.

32. Of $3x^5 - 10x^3 + 15x - 8$ and $x^4 - 2x^2 + 1$ is $x^2 - 2x + 1$.

33. Of $15x^4 - 9x^3 + 47x^2 - 21x + 28$ and $20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4$ is $5x^2 - 3x + 4$.

34. Of $x^4 + ax^3 - 9a^2x^2 + 11a^3x - 4a^4$ and $x^4 - ax^3 - 3a^2x^2 + 5a^3x - 2a^4$ is $(x-a)^3$.

35. Of $4a^4 - 4a^2b^2 + 4ab^3 - b^4$ and $6a^4 + 4a^3b - 9a^2b^2 - 3ab^3 + 2b^4$ is $2a^2 + 2ab - b^2$.

36. Of $x^5 - bx^4 - b^4x + b^5$ and $x^4 - bx^3 - b^2x^2 + b^3x$ is $(x-b)(x^2-b^2)$.

37. Of $x^4 - y^4$ and $x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4$ is $x^3 + x^2y + xy^2 + y^3$.

38. Of $a^2 + b^2 + c^2 + 2(ab + ac + bc)$ and $a^2 - b^2 - c^2 - 2bc$ is $a + b + c$.

39. Of $a^2 - 3ab + ac + 2b^2 - 2bc$ and $a^2 + b^2 - c^2 + 2bc$ is $a - b + c$.

40. Of $a^4 + b^4 - c^4 + 2a^3b + 2a^2b^2 + 2ab^3 + 2abc^2$ and $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$ is $a^2 + 2ab + b^2 - c^2$.

41. Of $x^5 - a^2x^3 + a^3x^2 - 2a^4x + a^5$ and $x^6 + 2a^3x^3 - 2a^4x^2 + 2a^5x - a^6$ is $x^4 + ax^3 + a^3x - a^4$.

42. Of $3x^2 - (4a + 2b)x + a^2 + 2ab$ and $x^3 - (2a + b)x^2 + (a + 2b)ax - a^2b$ is $x - a$.

43. Of $x^4 - 2a(a-b)x^2 + (a^2 + b^2)(a-b)x - a^2b^2$ and $x^4 - (a-b)x^3 + (a-b)b^2x - b^4$ is $x^2 - (a-b)x + b^2$.

44. Of $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$ is $2x^3 - 4x^2 + x - 1$.

45. Of $15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$ and $6a^3 + 19a^2b + 8ab^2 - 5b^3$ is $3a^2 + 2ab - b^2$.

46. Of $x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9$ and $6x^5 + 20x^4 - 12x^3 - 48x^2 + 22x + 12$ is $x^3 + x^2 - 5x + 3$.

47. Of $x^5 - bx^4 - 2a^2x^3 + a^3x^2 + (a + 2b)a^2bx - 2a^3b^2$ and $x^4 - (a^2 + b^2)x^2 + a^2b^2$ is $x^2 - (a + b)x + ab$.

48. Of $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ and $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$ is $2a + 2b + c$.

49. Of $e^{2x}x^3 + e^{2x} - x^3 - 1$ and $e^{2x}x^2 + 2e^x x^2 - e^{2x} - 2e^x + x^2 - 1$ is $(x + 1)(e^x + 1)$.

50. Of $6x^2 + 4x^2y$, $2ax^4 - 8bx^2y^2$ and $4cx^5 + 12dx^4y$ is $2x^2$.

51. Of $a^3 + 5a^2x + 7ax^2 + 3x^3$, $a^3 + 3a^2x - ax^2 - 3a^3$ and $a^3 + a^2x - 5ax^2 + 3x^3$ is $a + 3x$.

52. Of $x^4 - ax^3 + (b - 1)x^2 + ax - b$, $x^4 - bx^3 + (a - 1)x^2 + bx - a$ and $x^4 - (a - 1)x^3 - (a - b)x^2 + (b - 1)x - 1$ is $x + 1$.

## II. COMMON MULTIPLES.

1. The least common multiple of $axy$ and $a(xy - y^2)$ is $ax^2y - axy^2$.

2. Of $ab + ad$ and $ab - ad$ is $ab^2 - ad^2$.

3. Of $x^3 + 1$ and $(x + 1)^2$ is $x^4 + x^3 + x + 1$.

4. Of $x^3 - 7x^2 + 16x - 12$ and $3x^3 - 14x^2 + 16x$ is $3x^5 - 29x^4 + 104x^3 - 164x^2 + 96x$.

5. Of $12x^2 - 17ax + 6a^2$ and $9x^2 + 6ax - 8a^2$ is $36x^3 - 3ax^2 - 50a^2x + 24a^3$.

6. Of $a^3+2a^2b-ab^2-2b^3$ and $a^3-2a^2b-ab^2+2b^3$ is $a^4-5a^2b^2+4b^4$.

7. Of $x^4-ax^3y+x^2y^2-axy^3$ and $x^2-axy+xy-ay^2$ is $x^5-(a-1)x^4y-(a-1)x^3y^2-(a-1)x^2y^3-axy^4$.

8. Of $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$ and $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5$ is $x^6-a^6$.

9. Of $a^4-3a^3b+4a^2b^2-3ab^3+b^4$ and $a^4+a^3b+ab^3+b^4$ is $a^6-a^5b-a^4b^2+2a^3b^3-a^2b^4-ab^5+b^6$.

10. Of $x^4-(a^2+b^2)x^2+a^2b^2$ and $x^4-(a+b)^2x^2+2(a+b)abx-a^2b^2$ is $x^6+(a+b)x^5-(a^2+ab+b^2)x^4-(a+b)(a^2+b^2)x^3+(a^2+ab+b^2)abx^2+(a+b)a^2b^2x-a^3b^3$.

11. Of $x$, $ax$ and $a\pm x$ is $a^2x\pm ax^2$.

12. Of $a^2-b^2$, $(a-b)^2$ and $a^3+b^3$ is $a^5-2a^4b+a^3b^2-a^2b^3-2ab^4+b^5$.

13. Of $a^2-ax+x^2$, $a^2+ax+x^2$, $a^3-x^3$ and $a^3+x^3$ is $a^6-x^6$.

## CHAP. IV.

### MISCELLANEOUS EXAMPLES.

1. $$xy=\frac{xy^2(a+y)}{ay+y^2}=\frac{axy^2+xy^3}{ay+y^2}=\frac{axy+xy^2}{a+y}.$$

2. $$\frac{a^3+a^2b-ab^2-b^3}{a-b}=a^2+2ab+b^2=(a+b)^2.$$

3. $$\frac{x^8-x^2}{x^2-1}=x^2\left(\frac{x^6-1}{x^2-1}\right)=x^6+x^4+x^2,\ \frac{a^2x^2-b^2}{ax-b}=ax+b,$$

and $$\frac{a^3x^3-b^3}{ax-b}=a^2x^2+abx+b^2.$$

4. $$a+x+\frac{a^2-ax}{x}=\frac{a^2+x^2}{x}\text{ and }2a-x+\frac{(a-x)^2}{x}=\frac{a^2}{x}.$$

5. $a^2 + \left(\frac{2ax}{x^2-1}\right)^2 = \left(\frac{x^2+1}{x^2-1}\right)^2 a^2$, $\left(\frac{x^2+1}{2x}\right)^2 a^2 - a^2 =$

$\left(\frac{x^2-1}{2x}\right)^2 a^2$ and $(a-x)^2 + \left(\frac{a^2+x^2}{a+x}\right)^2 = \frac{2(a^4+x^4)}{(a+x)^2}$.

6. $1 - \frac{a^2+b^2-c^2}{2ab} = \frac{c^2-(a-b)^2}{2ab}$ and $b^2 - \left(\frac{b^2+c^2-a^2}{2c}\right)^2$

$$= \frac{\{(b+c)^2-a^2\}\{a^2-(b-c)^2\}}{4c^2}.$$

7. $1 + \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} = \frac{(a+b)^2-(c-d)^2}{2(ab+cd)}$,

and $1 - \frac{a^2+b^2-c^2-d^2}{2(ab+cd)} = \frac{(c+d)^2-(a-b)^2}{2(ab+cd)}$.

8. $$1 - \left\{\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$$

$$= \frac{\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}}{4(ab+cd)^2}$$

$$= \frac{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}{4(ab+cd)^2}.$$

9. $a - x + \frac{4ax}{a-x} = \frac{(a+x)^2}{a-x}$, $a + x - \frac{4ax}{a+x} = \frac{(a-x)^2}{a+x}$.

10. $(a-x)^2 + \frac{6a^2x+2x^3}{a-x} = \frac{(a+x)^3}{a-x}$, $(a+x)^2 - \frac{6a^2x+2x^3}{a+x}$

$= \frac{(a-x)^3}{a+x}$ and $a^2 - 6ax + 17x^2 - \frac{16x^3(2a+x)}{(a+x)^2} = \frac{(a-x)^4}{(a+x)^2}$.

11. $\frac{a^3+x^3}{a-x} = a^2+ax+x^2+\frac{2x^3}{a-x}$, $\frac{a^3-x^3}{a+x} = a^2-ax+x^2-\frac{2x^3}{a+x}$.

12. $\frac{a^2}{a+x} = a - x + \frac{x^2}{a+x}$ and $\frac{x^2}{x-a} = x + a + \frac{a^2}{x-a}$.

13. $\dfrac{a^5}{a^2-x^2}=a^3+ax^2+\dfrac{ax^4}{a^2-x^2}$, $\dfrac{x^5}{a^2-x^2}=-x^3-a^2x+\dfrac{a^4x}{a^2-x^2}$, $\dfrac{a^5x^5}{a^2-x^2}=a^3x^5+ax^7+\dfrac{ax^9}{a^2-x^2}=-a^5x^3-a^7x+\dfrac{a^9x}{a^2-x^2}$.

14. $\dfrac{x^m}{x-a}=x^{m-1}+ax^{m-2}+a^2x^{m-3}+\text{\&c.}+a^{m-2}x+a^{m-1}+\dfrac{a^m}{x-a}$,

$$\frac{x^m}{x^2+a^2}=x^{m-2}-a^2x^{m-4}+a^4x^{m-6}-\text{\&c.}\mp a^{m-4}x^2\pm a^{m-2}\mp\frac{a^m}{x^2+a^2}.$$

15. $\dfrac{x^2+px+q}{x+a}=x-a+p+\dfrac{a^2-pa+q}{x+a}$, $\dfrac{x^3-px^2+px-1}{x-a}$

$$=x^2+(a-p)x+a^2-pa+p+\frac{a^3-pa^2+pa-1}{x-a}.$$

16. $\dfrac{x^{mn-1}}{x^n-a^n}=x^{mn-n-1}+a^nx^{mn-2n-1}+a^{2n}x^{mn-3n-1}+\text{\&c.}$

$$+a^{(m-2)n}x^{n-1}+\frac{a^{(m-1)n}x^{n-1}}{x^n-a^n}.$$

17. $\dfrac{a^2+2ab+b^2}{a^2-b^2}=\dfrac{a+b}{a-b}$ and $\dfrac{a^2-2ab+b^2}{a^2-b^2}=\dfrac{a-b}{a+b}$.

18. $\dfrac{a^3+x^3}{(a+x)^2}=\dfrac{a^2-ax+x^2}{a+x}$ and $\dfrac{a^3-x^3}{(a-x)^2}=\dfrac{a^2+ax+x^2}{a-x}$.

19. $\dfrac{(a-b)^3(c+x)^2}{(a^2-b^2)(c^3+x^3)}=\dfrac{(a-b)^2(c+x)}{(a+b)(c^2-cx+x^2)}$, $\dfrac{a^3b^3+c^3x^3}{a^2b^2-c^2x^2}$

$$=\frac{a^2b^2-abcx+c^2x^2}{ab-cx} \text{ and } \frac{ac+ad+bc+bd}{ae+af+be+bf}=\frac{c+d}{e+f}.$$

20. $\dfrac{x^2+(a-b)x-ab}{x^2-(a+b)x+ab}=\dfrac{x+a}{x-a}$, $\dfrac{6a^3-6a^2y+2ay^2-2y^3}{12a^2-15ay+3y^2}$

$$=\frac{6a^2+2y^2}{12a-3y} \text{ and } \frac{x^5+5bx^4-b^2x^2-5b^3x}{x^4-3bx^3-b^2x-3b^3}=\frac{x^2+5bx}{x+3b}.$$

21. $\dfrac{4a^4-4a^2b^2+4ab^3-b^4}{6a^4+4a^3b-9a^2b^2-3ab^3+2b^4}=\dfrac{2a^2-2ab+b^2}{3a^2-ab-2b^2}$.

22. $\dfrac{(x+y)^3 - z^2(x+y)}{4y^2z^2 - (x^2-y^2-z^2)^2} = \dfrac{x+y}{z^2-(x-y)^2}$ and

$$\frac{a^2+b^2+c^2+2ab-2ac-2bc}{a^2-b^2-c^2+2bc} = \frac{a+b-c}{a-b+c}.$$

23. $\dfrac{a^2 - acx + (ac - b^2 + bc)x^2 - bcx^3}{a^2 + abx + (ac - c^2 + bc)x^2 + c^2x^3} = \dfrac{a-bx}{a+cx}.$

24. $\dfrac{a^2(b^2-c^2) - ab(2b^2+bc-c^2) + b^3(b+c)}{a^2(b+c)^2 - ab(2b^2+3bc+c^2) + b^3(b+c)} = \dfrac{a(b-c)-b^2}{a(b+c)-b^2}.$

25. $\dfrac{a+x}{a-x}$ and $\dfrac{a-x}{a+x}$ are equal to $\dfrac{(a+x)^2}{a^2-x^2}$ and $\dfrac{(a-x)^2}{a^2-x^2}.$

26. $1+\dfrac{x-y}{x+y}$, $2+\dfrac{(x-y)^2}{2xy}$ and $\dfrac{(x+y)^3}{xy(x-y)}-3$

are equal to $\dfrac{4x^3y-4x^2y^2}{2xy(x^2-y^2)}$, $\dfrac{x^4+2x^3y-2xy^3-y^4}{2xy(x^2-y^2)}$

and $\dfrac{2x^4+2x^3y+12x^2y^2+14xy^3+2y^4}{2xy(x^2-y^2)}.$

27. $\dfrac{a+b}{2}+\dfrac{a-b}{2}=a$ and $\dfrac{a+b}{2}-\dfrac{a-b}{2}=b.$

28. $\dfrac{a^2}{2}(a^2-b^2)-\dfrac{1}{4}(a^4-b^4)=\dfrac{(a^2-b^2)^2}{4}$, $\dfrac{1}{a^m+1}$

$=\dfrac{1}{a^m-1}-\dfrac{2}{a^{2m}-1}$ and $\dfrac{m}{a^m-1}+\dfrac{m}{a^{-m}-1}=-m.$

29. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\dfrac{bcx+acy+abz}{abc}$,

and $\dfrac{a}{x^n}+\dfrac{b}{x^{n-m}}-\dfrac{c}{x^{n-2m}}=\dfrac{a+bx^m-cx^{2m}}{x^n}.$

30. $\dfrac{(x+a)^2}{x-a}+\dfrac{x^2}{x-2a}=\dfrac{2x^3-ax^2-3a^2x-2a^3}{x^2-3ax+2a^2}$,

and $\dfrac{(x+a)^2}{x-a}-\dfrac{x^2}{x-2a}=\dfrac{a(x^2-3ax-2a^2)}{x^2-3ax+2a^2}.$

31. $$\frac{1+x}{1+x+x^2}+\frac{1-x}{1-x+x^2}=\frac{2}{1+x^2+x^4},$$

and $$\frac{1+x}{1+x+x^2}-\frac{1-x}{1-x+x^2}=\frac{2x^3}{1+x^2+x^4}.$$

32. $$\frac{a}{a+b}+\frac{b}{a-b}=\frac{a^2+b^2}{a^2-b^2}=\frac{a}{a-b}-\frac{b}{a+b},$$

and $$\frac{a}{a+c}-\frac{b}{b+c}=\frac{ac-bc}{(a+c)(b+c)}=\frac{a}{b+c}-\frac{c}{a+c}.$$

33. $$\frac{1}{x-1}+\frac{x-1}{x^2+x+1}=\frac{2x^2-x+2}{x^3-1},$$

and $$\frac{1}{x+1}-\frac{x+1}{x^2-x+1}=-\frac{3x}{x^3+1}.$$

34. $$\frac{3}{x+1}+\frac{x+1}{x^2+1}=\frac{2(2x^2+x+2)}{(x+1)(x^2+1)},$$

and $$\frac{3}{x+1}-\frac{x+1}{x^2+1}=\frac{2(x^2-x+1)}{(x+1)(x^2+1)}.$$

35. $$\frac{2x}{x^2+b}-\frac{2x}{x^2+a}=\frac{2(a-b)x}{(x^2+a)(x^2+b)},$$

and $$\frac{2x}{x^2+a}+\frac{2x}{x^2+b}=\frac{4x^3+2(a+b)x}{(x^2+a)(x^2+b)}.$$

36. $$\frac{a^2+b^2}{a^2-b^2}+\frac{a-b}{a+b}=\frac{2(a^2-ab+b^2)}{a^2-b^2},$$

and $$2+\frac{a^2+b^2}{a^2-b^2}-\frac{a-b}{a+b}=\frac{2(a^2+ab-b^2)}{a^2-b^2}.$$

37. $$\frac{2a^2-2a+1}{a^2-a}+\frac{a}{a-1}=\frac{3a^2-2a+1}{a^2-a},$$

and $$\frac{2a^2-2a+1}{a^2-a}-\frac{a}{a-1}=\frac{a-1}{a}=1-\frac{1}{a}.$$

38. $$\frac{1}{x^2+1}+\frac{2x+1}{(x^2+1)^2}-\frac{2x-1}{(x^2+1)^3}=\frac{x^4+2x^3+3x^2+3}{(x^2+1)^3}.$$

39. $$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}=\frac{6x^2-22x+18}{x^3-6x^2+11x-6},\ \frac{1}{(x-1)^3}+\frac{2}{(x-1)^2}+\frac{3}{x-1}=\frac{3x^2-4x+2}{(x-1)^3},\ \frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}=\frac{2a^2}{x(a^2-x^2)}$$ and $$\frac{1}{4(1+x)}+\frac{1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1}{1-x^4}.$$

40. $$\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{(x+2)^2}=\frac{1}{x^3+5x^2+8x+4},\ \frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}=\frac{1+x^2}{x(1-x^2)},\ \frac{3}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}=\frac{1-5x}{(1-x)(1+x)^2}.$$

41. $$\frac{a^2}{x+a}+\frac{b^2-2ab}{x+b}+\frac{(a-b)b^2}{(x+b)^2}=\frac{(a-b)^2x^2}{(x+a)(x+b)^2}.$$

42. $$\frac{1}{x}-\frac{1}{(x+1)^2}-\frac{2}{x+1}+\frac{x}{1+x+x^2}=\frac{1}{x(1+x)^2(1+x+x^2)}.$$

43. $$\frac{1}{3x-1}+\frac{2}{x-1}-\frac{1}{x}=\frac{4x^2+x-1}{3x^3-4x^2+x}$$ and $$\frac{1}{x-2}+\frac{1}{x+1}-\frac{1}{x-3}-\frac{1}{(x+1)^2}=\frac{x^3-6x^2+4x-1}{x^4-3x^3-3x^2+7x+6}.$$

44. $$\frac{1}{3(1+x)}+\frac{2-x}{3(1-x+x^2)}=\frac{1}{1+x^3}$$ and $$\frac{1}{1+x}-\frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}=\frac{2x^2+21x+13}{4x^4-5x^2+1}.$$

45. $$1-\frac{2x^2}{a^2}+\frac{2x^4}{a^2(a^2+x^2)}=\frac{a^2-x^2}{a^2+x^2}$$ and $$\frac{1}{(1+x)^2}+\frac{3}{1+x}+\frac{5}{(1+2x)^2}-\frac{6}{1+2x}=\frac{3x^2+5x+3}{(1+x)^2(1+2x)^2}.$$

46. $\frac{b}{d}+\frac{ad-bc}{d(c+dx)}=\frac{a+bx}{c+dx}$ and $\frac{2x^2+x+1}{x^3}+\frac{1}{2(1-x)^2}$

$$+\frac{7}{4(1-x)}-\frac{1}{4(1+x)}=\frac{1}{x^3(1-x^2)(1-x)}.$$

47. $\frac{a}{c}+\frac{(ad-bc)x}{c(c-dx)}=\frac{a-bx}{c-dx}$ and $\frac{1}{x}+\frac{1}{1+x}+\frac{1}{2-x}$

$$+\frac{2}{1-2x}+\frac{14}{(1-2x)^2}=\frac{x^2-28x-2}{x(x+1)(x-2)(2x-1)^2}.$$

48. $\frac{1}{1+x}-\frac{6(x-1)}{x^2}+\frac{2(x-1)}{x^2+2}+\frac{6(x-1)}{(x^2+1)^2}+\frac{3(x-1)}{x^2+1}$

$=\frac{12}{x^2(x+1)(x^2+2)(x^2+1)^2}$ and $\frac{2}{x}+\frac{1}{(x-1)^2}-\frac{3}{4(x-1)}$

$$-\frac{1}{2(x+1)^2}-\frac{5}{4(x+1)}=\frac{x^3+x^2+2}{x^5-2x^3+x}.$$

49. $\frac{1}{3(1-x)}+\frac{2+x}{3(1+x+x^2)}=\frac{1}{1-x^3}$ and $\frac{1}{x-1}+\frac{2}{(x+1)^2}$

$$+\frac{9}{x+1}-\frac{8(x^2-x+1)}{x^3}-\frac{2(x+1)}{x^2+1}=\frac{8}{x^8+x^7-x^4-x^3}.$$

50. $\frac{1}{x-6}-\frac{1}{x-5}=\frac{1}{x^2-11x+30}$ and $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}$

$$+\frac{b}{c}+\frac{c}{b}=\frac{(a+b+c)(ab+ac+bc)}{abc}-3.$$

51. $\frac{a+b}{a+x}+\frac{a-b}{a-x}=\frac{2(a^2-bx)}{a^2-x^2}$ and $\frac{a^2}{(a-b)^2(x+a)}$

$$+\frac{b^2-2ab}{(a-b)^2(x+b)}+\frac{b^2}{(a-b)(x+b)^2}=\frac{x^2}{(x+a)(x+b)^2}.$$

52. $\frac{a+b}{a+x}-\frac{a-b}{a-x}+\frac{(a-b)^2}{a^2-x^2}=\frac{a^2-2ax+b^2}{a^2-x^2}$ and $(a-b)$

$$\left\{\frac{1}{(x+a)^2}+\frac{1}{(x+b)^2}\right\}+2\left\{\frac{1}{x+a}-\frac{1}{x+b}\right\}=\frac{(a-b)^3}{(x+a)^2(x+b)^2}.$$

53. $$(a+b)\left\{\frac{1}{x+b}-\frac{1}{x+a}\right\}-(a-b)\left\{\frac{a}{(x+a)^2}+\frac{b}{(x+b)^2}\right\}$$

$$=\frac{(a-b)^3 x}{(x+a)^2(x+b)^2},\quad + 2ab\left\{\frac{1}{x+a}-\frac{1}{x+b}\right\}+(a-b)$$

$$\left\{\frac{a^2}{(x+a)^2}+\frac{b^2}{(x+b)^2}\right\}=\frac{(a-b)^3x^2}{(x+a)^2(x+b)^2} \text{ and } \frac{a^2(a-3b)}{x+a}-$$

$$\frac{b^2(b-3a)}{x+b}-(a-b)\left\{\frac{a^3}{(x+a)^2}+\frac{b^3}{(x+b)^2}\right\}=\frac{(a-b)^3x^3}{(x+a)^2(x+b)^2}.$$

54. $$\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}$$

$$+\frac{1}{(a-c)(b-c)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}.$$

55. $$-\frac{a}{(a-b)(a-c)(x+a)}+\frac{b}{(a-b)(b-c)(x+b)}$$

$$-\frac{c}{(a-c)(b-c)(x+c)}=\frac{x}{(x+a)(x+b)(x+c)}.$$

56. $$\frac{a^2}{(a-b)(a-c)(x+a)}-\frac{b^2}{(a-b)(b-c)(x+b)}$$

$$+\frac{c^2}{(a-c)(b-c)(x+c)}=\frac{x^2}{(x+a)(x+b)(x+c)}.$$

57. $$\frac{a}{bx}\times\frac{cx}{d}=\frac{ac}{bd} \text{ and } \frac{5ax}{bcy}\times\frac{xy+y^2}{x^2-xy}=\frac{5ax+5ay}{bcx-bcy}.$$

58. $$\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}\times\frac{a^2-ax+x^2}{a+x}=\frac{a^4+a^2x^2+x^4}{a^4-x^4}$$

$$\text{and } \frac{x^2-9x+20}{x^2-6x}\times\frac{x^2-13x+42}{x^2-5x}=\frac{x^2-11x+28}{x^2}.$$

59. $$\frac{a+b}{a-b}\times\frac{a-b}{a+b}=1,\quad \frac{a^2+2ab+b^2}{cd-d^2}\times\frac{d^2}{a+b}=\frac{(a+b)d}{c-d}$$

$$\text{and } \frac{4ax}{3by}\times\frac{a^2-x^2}{c^2-x^2}\times\frac{bc+bx}{a^2-ax}=\frac{4ax+4x^2}{3cy-3xy}.$$

60. $\dfrac{a^2-b^2}{x+y} \times \dfrac{x^2-y^2}{a-b} \times \dfrac{a^2}{x-y} = (a+b)a^2$ and

$$\left\{a+\frac{ax}{a-x}\right\} \times \left\{a-\frac{ax}{a+x}\right\} \times \frac{a^2-x^2}{a^2+x^2} = \frac{a^4}{a^2+x^2}.$$

61. $\left\{\dfrac{a}{a-b}+\dfrac{b}{a+b}\right\}\left\{\dfrac{a}{a-b}-\dfrac{b}{a+b}\right\} = \dfrac{a^3(a+2b)-b^3(b-2a)}{(a^2-b^2)^2}.$

62. $\left(\dfrac{a^2}{x^2}-\dfrac{ab}{2xy}+\dfrac{b^2}{y^2}\right) \times \left(\dfrac{3a^2}{x^2}-\dfrac{2ab}{5xy}+\dfrac{b^2}{y^2}\right)$

$$= \frac{3a^4}{x^4} - \frac{19a^3b}{10x^3y} + \frac{21a^2b^2}{5x^2y^2} - \frac{9ab^3}{10xy^3} + \frac{b^4}{y^4}.$$

63. $\{15a^{-6}b^2 - 7a^{-5}b^4 + 6a^{-4}b^6\} \times \{8a^{-2}b^2 - 3a^{-1}b^4\}$
$= 120a^{-8}b^4 - 101a^{-7}b^6 + 69a^{-6}b^8 - 18a^{-5}b^{10}.$

64. $\{13a^{-5}b + 10a^{-2}b^2 - 4ab^3\} \times \{6a^{-3}b^2 - 18b^3 - 7a^3b^4\}$
$= 78a^{-8}b^3 - 174a^{-5}b^4 - 295a^{-2}b^5 + 2ab^6 + 28a^4b^7.$

65. $\left(\dfrac{x}{a}+\dfrac{a}{x}\right)\left(\dfrac{y}{b}+\dfrac{b}{y}\right) + \left(\dfrac{x}{a}-\dfrac{a}{x}\right)\left(\dfrac{y}{b}-\dfrac{b}{y}\right) = \dfrac{2xy}{ab} + \dfrac{2ab}{xy},$

and $\left(\dfrac{x}{a}+\dfrac{a}{x}\right)\left(\dfrac{y}{b}+\dfrac{b}{y}\right) - \left(\dfrac{x}{a}-\dfrac{a}{x}\right)\left(\dfrac{y}{b}-\dfrac{b}{y}\right) = \dfrac{2bx}{ay} + \dfrac{2ay}{bx}.$

66. $\left(\dfrac{x}{a}-\dfrac{a}{x}\right)\left(\dfrac{y}{b}+\dfrac{b}{y}\right) + \left(\dfrac{x}{a}+\dfrac{a}{x}\right)\left(\dfrac{y}{b}-\dfrac{b}{y}\right) = \dfrac{2xy}{ab} - \dfrac{2ab}{xy},$

and $\left(\dfrac{x}{a}-\dfrac{a}{x}\right)\left(\dfrac{y}{b}+\dfrac{b}{y}\right) - \left(\dfrac{x}{a}+\dfrac{a}{x}\right)\left(\dfrac{y}{b}-\dfrac{b}{y}\right) = \dfrac{2bx}{ay} - \dfrac{2ay}{bx}.$

67. $\dfrac{a}{c} \div \dfrac{b}{c} = \dfrac{a}{b}$, $\dfrac{ax}{by} \div \dfrac{cx}{dy} = \dfrac{ad}{bc}$ and $\dfrac{3a^2x}{2b^3} \div \dfrac{ab}{4x} = \dfrac{6ax^2}{b^4}.$

68. $\dfrac{2a(1-x^2)^2}{cy} \div \dfrac{(1-x)(1+x)^2}{y^3} = \dfrac{2ay^2(1-x)}{c}.$

69. $\dfrac{a+b}{2} \div \dfrac{a-b}{2} = \dfrac{a+b}{a-b}$, $\dfrac{a^2-b^2}{a+2b} \div \dfrac{a-b}{3a+6b} = 3(a+b),$

$$\frac{a+b}{a^2+2b^2} \div \frac{a^2-2b^2}{a-b} = \frac{a^2-b^2}{a^4-4b^4}, \quad \frac{x^2+xy}{x-y} \div \frac{x^4-y^4}{(x-y)^2} = \frac{x}{x^2+y^2}$$

$$\text{and } \frac{3ax+x^2}{a^3-x^3} \div \frac{x}{a-x} = \frac{3a+x}{a^2+ax+x^2}.$$

70. $$\left\{a^2 - \frac{4a^2x - 3ax^2 + x^3}{a+x}\right\} \div \left\{a^2 + \frac{4a^2x + 3ax^2 + x^3}{a-x}\right\}$$

$$= \left(\frac{a-x}{a+x}\right)^4, \quad \left\{x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)\right\} \div \left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)^2.$$

71. $$\left\{x^3 + \frac{1}{x^3} + 5\left(x + \frac{1}{x}\right)\right\} \div \left(x + \frac{1}{x}\right) = x^2 + 4 + \frac{1}{x^2}.$$

72. $$\left\{\frac{a^4}{b^6} - \frac{4c^6d^8}{b^{10}} + \frac{14c^5d^4}{a^4b^8} - \frac{49c^4}{4a^8b^6}\right\} \div \left\{\frac{a^2}{b^3} - \frac{2c^3d^4}{b^5} + \frac{7c^2}{2a^4b^3}\right\}$$

$$= \frac{a^2}{b^3} + \frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^3}.$$

73. $(a+x)^2(a-y)^{-3} \div (a+x)^{-4}(a-y)^{-7} = (a+x)^6(a-y)^4.$

74. $$\{-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8\} \div (2a^{-3}x^3 - 3ax^4)$$
$$= -a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4.$$

75. $$\left(\frac{a}{bx}\right)^3 = \frac{a^3}{b^3x^3} \text{ and } \left\{\frac{ax^3}{by^5}\right\}^{2m} = \left(\frac{a^2x^6}{b^2y^{10}}\right)^m = \frac{a^{2m}x^{6m}}{b^{2m}y^{10m}}.$$

76. $$\left(\pm\frac{2x}{3y}\right)^3 = \pm\frac{8x^3}{27y^3}, \quad \left(\frac{x}{a} + \frac{a}{x}\right)^3 = \frac{x^3}{a^3} + \frac{a^3}{x^3} + 3\left(\frac{x}{a} + \frac{a}{x}\right)$$

$$\text{and } \left(\frac{x}{y} - 1 - \frac{y}{x}\right)^3 = \frac{x^3}{y^3} - \frac{y^3}{x^3} - 3\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 5.$$

77. $$\left\{\left((a^{-m})^{-n}\right)^p\right\}^q = a^{mnpq} = \left\{\left((a^{-m})^{-n}\right)^{-p}\right\}^{-q}$$

$$\text{and } \left\{\left((a^m)^{-n}\right)^{-p}\right\}^{-q} = a^{-mnpq} = \left\{\left((a^{-m})^{-n}\right)^{-p}\right\}^q.$$

78. The square root of $\frac{a^2(x-y)^2}{b^2(x+y)^2} = \pm\left(\frac{ax-ay}{bx+by}\right)$,

of $\frac{a^2(a-x)^4}{(x-y)^6} = \pm\frac{a(a-x)^2}{(x-y)^3}$, of $\frac{a^2-2ab+b^2}{x^4+4ax^2+4a^2} = \frac{a-b}{x^2+2a}$.

79. Of $\frac{a^2}{b^2}+\frac{b^2}{a^2}\pm 2=\frac{a}{b}\pm\frac{b}{a}$ and of $\frac{a^2}{x^2}-\frac{4a}{3b}+\frac{4x^2}{9b^2}=\frac{a}{x}-\frac{2x}{3b}$.

80. Of $\frac{x}{a}\left(\frac{x}{a}-2\right)+\frac{b}{x}\left(\frac{b}{x}-2\right)+\frac{2b}{a}+1=\frac{x}{a}-1+\frac{b}{x}$.

81. Of $\frac{a^4}{b^2}+\frac{c^2}{a^4}+\frac{bc^2}{a^2}\left(b+\frac{2}{a}\right)+2c\left(a+\frac{1}{b}\right)=\frac{a^2}{b}+\frac{bc}{a}+\frac{c}{a^2}$.

82. Of $\frac{4x^2}{49y^2}-\frac{20x}{7y}+\frac{178}{7}-\frac{15y}{2x}+\frac{9y^2}{16x^2}=\frac{2x}{7y}-5+\frac{3y}{4x}$.

83. Of $\frac{4a^4}{9}+\frac{2a^3}{3b}+\frac{5a^2}{4b^2}+\frac{3a}{4b^3}+\frac{9}{16b^4}=\frac{2a^2}{3}+\frac{a}{2b}+\frac{3}{4b^2}$.

84. The cube root of $\frac{a^3b^6x^{-3m}}{27c^3y^{-6n}}=\frac{ab^2x^{-m}}{3cy^{-2n}}=\frac{ab^2y^{2n}}{3cx^m}$.

85. Of $\frac{a^3c^3x^6}{b^3}-\frac{3a^2cx^5}{b}+\frac{3abx^4}{c}-\frac{b^3x^3}{c^3}=\frac{acx^2}{b}-\frac{bx}{c}$.

86. Of $\frac{a^3y^3}{b^6c^3}+\frac{3a^2cy^4}{b^4d}-\frac{3a^3y^2}{b^4c^2}+\frac{3ac^5y^5}{b^2d^2}-\frac{6ac^2y^3}{b^2d}+\frac{3a^3y}{b^2c}$

$+\frac{c^9y^6}{d^3}-\frac{3ac^6y^4}{d^2}+\frac{3a^2c^3y^2}{d}-a^3=\frac{ay}{b^2c}+\frac{c^3y^2}{d}-a$.

87. Of $b^3+\frac{3a^2b^2x^{-2}}{2c^2}+\frac{3a^4bx^{-4}}{4c^4}+\frac{a^6x^{-6}}{8c^6}=b+\frac{a^2x^{-2}}{2c^2}$.

88. The 4th root of $\frac{a^8b^{20}c^4}{(a+f)^4h^{12}z^{16}}=\frac{a^2b^5c}{(a+f)h^3z^4}$ and of

$\frac{a^4(ax+x^2)^{-4}}{b^8(cx-x^2)^{-8}}=\frac{a(ax+x^2)^{-1}}{b^2(cx-x^2)^{-2}}=\frac{a(cx-x^2)^2}{b^2(ax+x^2)}=\frac{ax(c-x)^2}{b^2(a+x)}$.

89. The $5^{th}$ root of $\dfrac{a^{-5}(a+b)^5(2+x)^{-10}}{32c^5d^{-15}} = \dfrac{d^3(a+b)}{2ac(2+x)^2}$.

90. The $m^{th}$ root of

$$a^{mn}\,b^{mp}\,c^{-mq}\,d^{-mr} = a^n\,b^p\,c^{-q}\,d^{-r} = \frac{a^n\,b^p}{c^q\,d^r}.$$

91. $\dfrac{a}{x \pm 1} = \dfrac{a}{x} \pm \dfrac{a}{x^2} + \dfrac{a}{x^3} \pm \dfrac{a}{x^4}$ + &c. *in infinitum.*

92. $\dfrac{x+a}{x-b} = 1 + \dfrac{a+b}{x} + \dfrac{b(a+b)}{x^2} + \dfrac{b^2(a+b)}{x^3}$ + &c. *in infinitum.*

93. $\dfrac{a-x}{b+x} = \dfrac{a}{b} - \dfrac{a-b}{b^2}x + \dfrac{a-b}{b^3}x^2 - \dfrac{a-b}{b^4}x^3$ + &c. *in infinitum.*

94. If $x^2 = \dfrac{(c^2+d^2)ab-(a^2+b^2)cd}{ab-cd}$, prove that

$$1-\left(\frac{a^2+b^2-x^2}{2ab}\right)^2$$

$$= \frac{(a+b+c+d)(a+b-c-d)(a+c-b-d)(b+c-a-d)}{4(ab-cd)^2}.$$

95. What is the integer value of $x$, when $\frac{1}{4}(x+2)+\frac{1}{3}x$ is less than $\frac{1}{2}(x-4)+3$ and greater than $\frac{1}{2}(x+1)+\frac{1}{3}$? $x=5$.

## CHAP. V.

### MISCELLANEOUS EXAMPLES.

1. $2ax = \sqrt{4a^2x^2} = \sqrt[3]{8a^3x^3} = \sqrt[3]{\dfrac{8}{a^{-3}x^{-3}}} = \sqrt[4]{\dfrac{16a^4}{x^{-4}}} =$ &c.

2. $-\dfrac{4x}{3yz} = -\sqrt{\dfrac{16x^2}{9y^2z^2}} = \sqrt[3]{-\dfrac{64x^3}{27y^3z^3}} =$ &c.

3. $ax - by = \sqrt{a^2x^2 - 2abxy + b^2y^2}$

$= \sqrt[3]{a^3x^3 - 3a^2bx^2y + 3ab^2xy^2 - b^3y^3} =$ &c.

4. $\sqrt{a^2x^4y^6} = \pm ax^2y^3$, $\sqrt{9a^2(a-x)^2} = 3a(a-x)$

and $\sqrt[3]{\dfrac{64a^3(x^2-y^2)^6}{27b^3}} = \dfrac{4a(x^2-y^2)^2}{3b}$.

5. $\sqrt[m]{\dfrac{x^m y^{2m}}{z^{3m}}} = \dfrac{xy^2}{z^3}$, $\sqrt{\dfrac{a^3(a-2b)+b^3(2a-b)}{a^2-b^2}} = a - b$,

and $\sqrt[3]{\dfrac{(ax-x^2)^3(2cx+x^2)^3y^3}{x^6(by-y^2)^9}} = \dfrac{(a-x)(2c+x)}{y^2(b-y)^3}$.

6. $2\sqrt{\dfrac{a}{2}} = \sqrt{2a}$, $a\sqrt{ax} = \sqrt{a^3x}$, $bc\sqrt{\dfrac{a}{bc}} = \sqrt{abc}$,

and $(a+x)\sqrt{a-x} = \sqrt{(a+x)(a^2-x^2)}$.

7. $(a+x)\sqrt{\dfrac{a-x}{a+x}} = \sqrt{a^2-x^2}$, $(a+x)\sqrt{\dfrac{1}{a^2-x^2}}$

$= \sqrt{\dfrac{a+x}{a-x}}$ and $(a-x)\sqrt{\dfrac{1}{a^2-x^2}} = \sqrt{\dfrac{a-x}{a+x}}$.

8. $(x-1)\sqrt[3]{\dfrac{x+1}{x-1}} = \sqrt[3]{(x^2-1)(x-1)}$, $(x+1)\dfrac{1}{\sqrt[3]{(x^2-1)^2}}$

$= \sqrt[3]{\dfrac{x+1}{(x-1)^2}}$ and $(x-1)\dfrac{1}{\sqrt[3]{(x^2-1)^2}} = \sqrt[3]{\dfrac{x-1}{(x+1)^2}}$.

9. $(x+1)\sqrt[3]{\dfrac{x-1}{x+1}} = \sqrt[3]{(x+1)(x^2-1)}$

and $(a+x)\sqrt[4]{\dfrac{a^2+x^2}{a^2-x^2}} = \sqrt[4]{\dfrac{(a^2+x^2)(a+x)^3}{a-x}}$.

10. $\sqrt{a^2xy} = a\sqrt{xy}$ and $\sqrt[3]{b^3(x-y)^2} = b\sqrt[3]{(x-y)^2}$.

11. $\sqrt{x^2yz^2} = xz\sqrt{y}$, $\sqrt{a^4(a^2-x^2)} = a^2\sqrt{a^2-x^2}$
and $\sqrt[3]{a^3(a^3-ax^2)} = a\sqrt[3]{a^3-ax^2}$.

12. $\sqrt{3a^2x+6abx+3b^2x} = (a+b)\sqrt{3x}$,
and $\sqrt[3]{4a^3x+12a^2x^2+12ax^3+4x^4} = (a+x)\sqrt[3]{4x}$.

13. $\sqrt{a^3-3ax^2+2x^3} = (a-x)\sqrt{a+2x}$,
and $\sqrt[3]{a^4-6a^2x^2-8ax^3-3x^4} = (a+x)\sqrt{a-3x}$.

14. $\sqrt{a^3x^3-3ab^2x-2b^3} = (ax+b)\sqrt{ax-2b}$,
and $\sqrt{x^3-\left(\frac{3x-1}{4}\right)} = \left(x-\frac{1}{2}\right)\sqrt{x+1}$.

15. $\sqrt{\frac{1}{x}-2a+a^2x} = \left(\frac{1}{x}-a\right)\sqrt{x}$,
and $\sqrt[3]{\frac{a}{x}-3a+3ax-ax^2} = (1-x)\sqrt[3]{\frac{a}{x}}$.

16. $\sqrt{\left(\frac{x^3-1}{x^2-1}\right)+3} = \frac{x+2}{\sqrt{x+1}}$,
and $\sqrt[3]{\left(\frac{x^3-1}{3x}\right)-(x-1)} = \frac{x-1}{3x}\sqrt[3]{9x^2}$.

17. $\sqrt{\frac{3a^3x^2}{bc^2}-\frac{5a^2x^3}{b^2c}} = \frac{ax}{bc}\sqrt{3ab-5cx}$,
and $\sqrt{\frac{8a^4}{27x^3}+\frac{32a^3}{27x^2}} = \frac{2a}{3x}\sqrt[3]{a+4x}$.

18. $\sqrt{\frac{a^3y+2a^2y+ay}{x^3-x^2y}} = \frac{a+1}{x(x-y)}\sqrt{ay(x-y)}$,
and $\sqrt[3]{\frac{(x+a)^2(x^2-a^2)}{(x^2-ax)y}} = \frac{x+a}{xy}\sqrt[3]{(xy)^2}$.

19. $\sqrt{a^{-4}x-2a^{-2}x^2+x^3}=\dfrac{1-a^2x}{a^2}\sqrt{x}$,

and $\sqrt[3]{\dfrac{(x+1)^{-4}(x-1)^{-2}}{x^2-x}}=\dfrac{1}{x^2-1}\sqrt[3]{\dfrac{1}{x^2+x}}$.

20. $\left(\dfrac{a}{b}\right)^{\frac{1}{2}}$ and $\left(\dfrac{b}{a}\right)^{\frac{3}{2}}=\sqrt{\dfrac{a^4}{a^3b}}$ and $\sqrt{\dfrac{b^4}{a^3b}}$.

21. $(a+b)^{\frac{3}{4}}$ and $(a-b)^{\frac{1}{2}}=\sqrt[4]{a^3+3a^2b+3ab^2+b^3}$ and $\sqrt[4]{a^2-2ab+b^2}$.

22. $\sqrt{a^2-x^2}$ and $\sqrt[3]{a^3+x^3}=\sqrt[6]{a^6-3a^4x^2+3a^2x^4-x^6}$ and $\sqrt[6]{a^6+2a^3x^3+x^6}$.

23. $(ax^2y)^{\frac{2}{3}}$, $(a+bx)^{\frac{1}{2}}$ and $\sqrt[3]{a^2-x^2}=\sqrt[6]{a^4x^8y^4}$, $\sqrt[6]{a^3+3a^2bx+3ab^2x^2+b^3x^3}$ and $\sqrt[6]{a^4-2a^2x^2+x^4}$.

24. $3\sqrt{x}+5\sqrt{x}=8\sqrt{x}$, $\sqrt{a^3b}+\sqrt{ab^3}=(a+b)\sqrt{ab}$ and $\sqrt{48a^4x}+\sqrt{27a^2x^3}+\sqrt{12x^5}=(4a^2+3ax+2x^2)\sqrt{3x}$.

25. $3a\sqrt{a^2b}+5\sqrt{16a^4b}=23a^2\sqrt{b}$,

$$\sqrt[3]{a^4x}+\sqrt[3]{ax^4}=(a+x)\sqrt[3]{ax}$$

and $x\sqrt{12a^2x}+2a\sqrt{27x^3}+3\sqrt{48a^2x^3}=20ax\sqrt{3x}$.

26. $\sqrt{\dfrac{7a^3}{x}}+\sqrt{\dfrac{63x^3}{a}}-\sqrt{112ax}=\left(\dfrac{a}{x}+\dfrac{3x}{a}-4\right)\sqrt{7ax}$.

27. $\sqrt{18a^5b^3}+\sqrt{50a^3b^3}=(3a^2b+5ab)\sqrt{2ab}$, $\sqrt[3]{16a^3x}+\sqrt[2]{4a^2x}+3\sqrt[4]{a^4x^2}+\sqrt[3]{128x^4}=(2a+4x)\sqrt[3]{2x}+5a\sqrt{x}$.

28. $7\sqrt{y}-3\sqrt{y}=4\sqrt{y}$, $\sqrt{18a^2b}-2a\sqrt{2b}=a\sqrt{2b}$

and $8\sqrt[3]{3a^3b}-a\sqrt[3]{\dfrac{8b}{9}}=\dfrac{22a}{3}\sqrt[3]{3b}$.

29. $\sqrt{45x^3}-\sqrt{80x^3}+\sqrt{5a^2x}=(a-x)\sqrt{5x}$, $\sqrt[4]{a^{13}x^4y}$ $-\sqrt[4]{a^9x^8y^5}+\sqrt[4]{ax^4y^9}=(a^3x-a^2x^2y+xy^2)\sqrt[4]{ay}$ and

$$\sqrt{\frac{a^2x}{y^3}}-\sqrt{\frac{a^2x^3}{b^2y}}-\sqrt{\frac{a^2b^2x}{c^2y}}=\left(\frac{a}{y}-\frac{ax}{b}-\frac{ab}{c}\right)\sqrt{\frac{x}{y}}.$$

30. $\sqrt[3]{\frac{27a^5x}{2b}}-\sqrt[3]{\frac{a^2x}{2b}}=(3a-1)\sqrt[3]{\frac{a^2x}{2b}}$, $\sqrt[3]{\frac{16a^4x}{3b^4}}$

$$+\sqrt[3]{\frac{2axy^3}{81bc^3}}-\sqrt[3]{\frac{ax^4}{24b^4}}=\left(\frac{2a}{b}+\frac{y}{3c}-\frac{x}{2b}\right)\sqrt[3]{\frac{2ax}{3b}} \text{ and}$$

$$b^2\sqrt{a^3x}-\frac{2b}{c}\sqrt{a^5x^3}+3x^2\sqrt{\frac{x}{a}}=\left(ab^2-\frac{2a^2bx}{c}+\frac{3x^2}{a}\right)\sqrt{ax}.$$

31. $3a^2x\sqrt[3]{(a-x)^4}-\sqrt[3]{\frac{8(ax-x^2)^7}{x}}+5ax^2\sqrt[3]{(a^4-a^3x)}$

$$=(3a^3x+4ax^3-2x^4)\sqrt[3]{a-x}.$$

32. $a\sqrt{b}\times b\sqrt{c}=ab\sqrt{bc}$, $a\sqrt{\frac{x}{y}}\times b\sqrt{\frac{y}{a}}=b\sqrt{ax}$,

$$\sqrt{\frac{a}{x}}\times\sqrt{\frac{x}{b}}\times\sqrt{\frac{b}{y}}=\sqrt{\frac{a}{y}},\quad \sqrt{\frac{2ax}{3b}}\times\sqrt{\frac{8ab}{5x}}=\frac{4a}{\sqrt{15}}$$

$$\text{and } -\sqrt[3]{\frac{2a^2x}{9b^2}}\times\sqrt[3]{\frac{4ax^2}{3b}}=-\frac{2ax}{3b}.$$

33. $\frac{a}{2b}\sqrt{a-x}\times(a-2b)\sqrt{a^2-ax}=\frac{(a^2-2ab)(a-x)}{2b}\sqrt{a}$,

$$\text{and } \frac{a}{b}\sqrt{ax}\times(b-x)\sqrt[3]{\frac{ax^2}{b}}=\frac{(b-x)a^{\frac{11}{6}}x^{\frac{7}{6}}}{b^{\frac{4}{3}}}.$$

34. $(a+\sqrt{ab}+b)(\sqrt{a}-\sqrt{b})=a\sqrt{a}-b\sqrt{b}$,

and $(\sqrt{x}+\sqrt[4]{xy}+\sqrt{y})\times(\sqrt{x}-\sqrt[4]{xy}+\sqrt{y})$
$=x+\sqrt{xy}+y.$

35. $(a+\sqrt{b}-d)\times(a-\sqrt{b})=a^2-ad-b+d\sqrt{b}$

and $(2a-3a\sqrt{d})\times(3c-2c\sqrt{d})=6ac(1+d)-13ac\sqrt{d}.$

36. $(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})=a-b,$

and $(a\sqrt{x}+b\sqrt{y})\times(a\sqrt{x}-b\sqrt{y})=a^2x-b^2y.$

37. $(a+\sqrt{x})\times(b+\sqrt{y})=ab+a\sqrt{y}+b\sqrt{x}+\sqrt{xy}$

and $(\sqrt{a}+b\sqrt[3]{x})\times(\sqrt{a}-b\sqrt[3]{x})=a-b^2\sqrt[3]{x^2}.$

38. $(a\sqrt[4]{x}-b\sqrt[4]{y})\times(c\sqrt[4]{x}+d\sqrt[4]{y})$
$=ac\sqrt{x}+(ad-bc)\sqrt[4]{xy}-bd\sqrt{y}$ and $(a+b\sqrt[6]{x})$
$\times(b+a\sqrt[6]{x})=ab(1+\sqrt[3]{x})+(a^2+b^2)\sqrt[6]{x}.$

39. $$\left(\sqrt{\frac{ax^2}{b^3}}+\sqrt{\frac{c}{d}}\right)\times\left(\frac{x}{b}\sqrt{\frac{a}{b}}-\sqrt{\frac{c}{d}}\right)=\frac{ax^2}{b^3}-\frac{c}{d},$$

and $$\left\{c\sqrt{\frac{a}{a+b}}+\sqrt{\frac{1}{b}}\right\}\times\left\{\frac{c}{d}\sqrt{a(a+b)}-\sqrt{\frac{b^5}{c^2}}\right\}$$
$$=\frac{ac^2}{d}-\frac{b^2}{c}+\left(\frac{c}{bd}-\frac{b^2}{a+b}\right)\sqrt{ab(a+b)}.$$

40. $\sqrt{a-\sqrt{b-\sqrt{3}}}\times\sqrt{a+\sqrt{b-\sqrt{3}}}=\sqrt{a^2-b+\sqrt{3}}.$

41. $(\sqrt{a}+\sqrt{b}+\sqrt{c})\times(\sqrt{a}+\sqrt{b}-\sqrt{c})\times(\sqrt{a}+\sqrt{c}$
$-\sqrt{b})\times(\sqrt{b}+\sqrt{c}-\sqrt{a})=-a^2-b^2-c^2+2(ab+ac+bc).$

42. $$\sqrt{ax}+\frac{b}{\sqrt{ax}}=\frac{ax+b}{\sqrt{ax}},\quad \left(\frac{a}{x}\right)^{\frac{1}{2}}-\left(\frac{x}{a}\right)^{\frac{3}{2}}=\frac{a^2-x^2}{a\sqrt{ax}}$$

and $$\left(\frac{a+x}{a-x}\right)^{\frac{3}{2}}-\left(\frac{a-x}{a+x}\right)^{\frac{5}{2}}=\frac{8ax(a^2+x^2)}{(a+x)(a^2-x^2)^{\frac{3}{2}}}.$$

43. $(a^2+3x^2)\sqrt{a^2-x^2}-(a^2+x^2)\dfrac{x^2}{\sqrt{a^2-x^2}}=\dfrac{a^4+a^2x^2-4x^4}{\sqrt{a^2-x^2}}.$

44. $4b\sqrt{ax+bx^2}-(a+2bx)^2(ax+bx^2)^{-\frac{1}{2}}=-\dfrac{a^2}{\sqrt{ax+bx^2}}.$

45. $4c\sqrt{a+bx+cx^2}-\dfrac{(b+2cx)^2}{\sqrt{a+bx+cx^2}}=\dfrac{4ac-b^2}{\sqrt{a+bx+cx^2}}.$

46. $\dfrac{1}{a-\sqrt{a^2-x^2}}+\dfrac{1}{a+\sqrt{a^2-x^2}}=\dfrac{2a}{x^2}$

and $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}=\dfrac{2}{x^2}\sqrt{a^2-x^2}.$

47. $\dfrac{1+x\sqrt{2}}{x+1}+\dfrac{1-x\sqrt{2}}{x-1}=\dfrac{(1-\sqrt{2})2x}{x^2-1},\ \dfrac{(1+\sqrt{5})x-4}{x^2-(1+\sqrt{5})x+2}$

$+\dfrac{(1-\sqrt{5})x-4}{x^2-(1-\sqrt{5})x+2}=\dfrac{2(x^3+6x-8)}{x^4-2x^3-4x+4}$ and

$$\frac{(1+\sqrt{5})x-4}{x^2-(1+\sqrt{5})x+2}-\frac{(1-\sqrt{5})x-4}{x^2-(1-\sqrt{5})x+2}=\frac{2x.\sqrt{5}(x^2-2)}{x^4-2x^3-4x+4}.$$

48. $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}+\dfrac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}$

$+\dfrac{1}{\sqrt{a}+\sqrt{c}-\sqrt{b}}+\dfrac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}$

$$=2\left\{\frac{(a-b-c)\sqrt{a}+(b-a-c)\sqrt{b}+(c-a-b)\sqrt{c}-2\sqrt{abc}}{a^2+b^2+c^2-2(ab+ac+bc)}\right\}.$$

49. $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}-\dfrac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}}$

$+\dfrac{1}{\sqrt{a}+\sqrt{c}-\sqrt{b}}-\dfrac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}}$

$$=2\left\{\frac{(a-b-c)\sqrt{a}-(b-a-c)\sqrt{b}+(c-a-b)\sqrt{c}+2\sqrt{abc}}{a^2+b^2+c^2-2(ab+ac+bc)}\right\}.$$

50. $(\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d})\times(\sqrt{a}+\sqrt{b}-\sqrt{c}+\sqrt{d})$
$=a+b-c-d+2\sqrt{ab}+2\sqrt{cd}.$

51. $\{(\sqrt{a}+\sqrt{b})x-\sqrt{ab}\}\times\{(\sqrt{a}-\sqrt{b})x-\sqrt{ab}\}$
$=(a-b)x^2-2ax\sqrt{b}+ab$ and $\{ab-(\sqrt{x}-\sqrt{y})^2\}$
$\times\{ab-(\sqrt{x}+\sqrt{y})^2\}=a^2b^2-2ab(x+y)+(x-y)^2.$

52. $(a^2+b\sqrt{b})\times(a^2-b\sqrt{b})=a^4-b^3$, $(\sqrt[8]{a}+\sqrt[8]{b})$
$\times(\sqrt[8]{a}-\sqrt[8]{b})\times(\sqrt[4]{a}+\sqrt[4]{b})\times(\sqrt{a}+\sqrt{b})=a-b,$
and $(a+\sqrt[4]{b})\times(a-\sqrt[4]{b})\times(a^2+\sqrt{b})=a^4-b.$

53. $(a^2+a\sqrt[3]{b}+b^{\frac{2}{3}})\times(a-\sqrt[3]{b})=a^3-b,$
and $(\sqrt[6]{a}+\sqrt[6]{b})\times(\sqrt[3]{a}+\sqrt[6]{ab}+\sqrt[3]{b})$
$\times(\sqrt[3]{a}-\sqrt[6]{ab}+\sqrt[3]{b})\times(\sqrt[6]{a}-\sqrt[6]{b})=a-b.$

54. $(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}})\times(x^{\frac{1}{4}}-y^{\frac{1}{4}})=x^{\frac{3}{4}}-y^{\frac{3}{4}},$
and $(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}})\times(x^{\frac{1}{3}}+y^{\frac{1}{3}})=x+y.$

55. $(a^{\frac{3}{2}}-ax^{\frac{3}{4}}+a^{\frac{1}{2}}x^{\frac{3}{2}}-x^{\frac{9}{4}})\times(a^{\frac{1}{2}}+x^{\frac{3}{4}})=a^2-x^3$ and
$(a^{\frac{5}{2}}+a^2x^{\frac{2}{3}}+a^{\frac{3}{2}}x^{\frac{4}{3}}+ax^2+a^{\frac{1}{2}}x^{\frac{8}{3}}+x^{\frac{10}{3}})\times(a^{\frac{1}{2}}-x^{\frac{2}{3}})=a^3-x^4.$

56. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})\times(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})=x^{\frac{1}{2}}y^{-\frac{1}{2}}+2+x^{-\frac{1}{2}}y^{\frac{1}{2}},$
and $(x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y)\times(x^{-1}+x^{-\frac{1}{2}}y^{-\frac{1}{2}}+y^{-1})$
$=xy^{-1}+2x^{\frac{1}{2}}y^{-\frac{1}{2}}+3+2x^{-\frac{1}{2}}y^{\frac{1}{2}}+x^{-1}y.$

57. $$\frac{\sqrt{x^2-a^2}-\sqrt{y^2-a^2}}{x-y}=\frac{x+y}{\sqrt{x^2-a^2}+\sqrt{y^2-a^2}},$$

and $$\frac{\sqrt{ax^2+bx+c}-\sqrt{ay^2+by+c}}{x-y}$$

$$=\frac{a(x+y)+b}{\sqrt{ax^2+bx+c}+\sqrt{ay^2+by+c}}.$$

58. $$\sqrt{\frac{a}{b}}\div\frac{a}{b}=\sqrt{\frac{b}{a}},\quad 5a\sqrt{ax}\div\frac{5}{2}\sqrt{bx}=2a\sqrt{\frac{a}{b}}$$

and $$4x\sqrt{a}\div\frac{3}{7}\sqrt[3]{ax}=\frac{28}{3}\sqrt[6]{ax^4}.$$

59. $$11x\sqrt{a^2-ax}\div x^2\sqrt{a^2x-ax^2}=\frac{11}{x\sqrt{x}},$$

and $$\frac{4a}{5b}\sqrt{a^3-x^3}\div\frac{3}{2}\sqrt{a^3-a^2x}=\frac{8}{15b}\sqrt{a^2+ax+x^2}.$$

60. $$ax\sqrt[4]{a^2-x^2}\div\sqrt[3]{a^2-ax}=a^{\frac{2}{3}}x(a+x)^{\frac{1}{4}}(a-x)^{-\frac{1}{12}},$$

$$\sqrt{\frac{a^3+x^3}{a-x}}\div\sqrt[3]{\frac{a^3-x^3}{a+x}}=\left(\frac{a+x}{a-x}\right)^{\frac{5}{6}}\sqrt[6]{\frac{(a^2-ax+x^2)^3}{(a^2+ax+x^2)^2}}.$$

61. $$\frac{a}{b}\sqrt[n]{\frac{ax}{cd}}\div\frac{c}{d}\sqrt[n]{\frac{by}{x}}=\frac{ad}{bc}\sqrt[n]{\frac{ax^2}{bcdy}},$$

and $$\frac{a}{x}\sqrt[m]{\frac{x}{y}}\div\frac{1}{y}\sqrt[n]{\frac{a}{x}}=a^{\frac{n-1}{n}}y^{\frac{m-1}{m}}x^{\frac{1}{m}+\frac{1}{n}-1}.$$

62. $$(a-b+c+2\sqrt{ac})\div(\sqrt{a}-\sqrt{b}+\sqrt{c})$$

$$=\sqrt{a}+\sqrt{b}+\sqrt{c},$$

and $$(a-b)\div(\sqrt[3]{a}-\sqrt[3]{b})=\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}.$$

63. $(a^3b-ab^2c)\div(a^2+a\sqrt{bc})=ab-b\sqrt{bc}$, and

$$(a\sqrt{b}+b\sqrt{ac}-\sqrt{abc}-bc)\div(\sqrt{a}+\sqrt{bc})=\sqrt{ab}-\sqrt{bc}.$$

64. $\left(\frac{x^2}{y^2}+\frac{x}{y}-\frac{x\sqrt{x}}{\sqrt{y}}+\frac{y\sqrt{y}}{\sqrt{x}}\right)\div\left(\sqrt{x}+\frac{y}{\sqrt{x}}\right)$

$=\frac{x\sqrt{x}}{y^2}-\frac{x}{\sqrt{y}}+\sqrt{y}$ and $\left(\frac{a^4}{x}+\frac{2a^2x^2}{\sqrt{ax}}+\frac{x^4}{a}\right)$

$\div\left(\frac{a^2}{\sqrt{x}}-a\sqrt{x}+\frac{x^2}{\sqrt{a}}\right)=\frac{a^2}{\sqrt{x}}+a\sqrt{x}+\frac{x^2}{\sqrt{a}}$.

65. $(x^{\frac{3}{2}}+2xy^{\frac{1}{2}}+2x^{\frac{1}{2}}y+y^{\frac{3}{2}})\div(x^{\frac{1}{2}}+y^{\frac{1}{2}})=x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$,

and $(x^{\frac{5}{2}}-x^2y^{\frac{1}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}+x^{\frac{1}{2}}y^2-y^{\frac{5}{2}})\div(x^2+xy+y^2)$

$=x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

66. $(a-2a^{\frac{1}{2}}x^{\frac{2}{3}}+x^{\frac{4}{3}})\div(a^{\frac{1}{2}}-2a^{\frac{1}{4}}x^{\frac{1}{3}}+x^{\frac{2}{3}})$

$=a^{\frac{1}{2}}+2a^{\frac{1}{4}}x^{\frac{1}{3}}+x^{\frac{2}{3}}$ and $(a^{\frac{3}{5}}-2a^{\frac{2}{5}}b^{\frac{2}{7}}+2a^{\frac{1}{5}}b^{\frac{4}{7}}-b^{\frac{6}{7}})$

$\div(a^{\frac{2}{5}}-a^{\frac{1}{5}}b^{\frac{2}{7}}+b^{\frac{4}{7}})=a^{\frac{1}{5}}-b^{\frac{2}{7}}$.

67. $(a^{\frac{7}{3}}-a^2b^{-\frac{2}{3}}-a^{\frac{1}{3}}b+b^{\frac{1}{3}})\div(a^{\frac{1}{3}}-b^{-\frac{2}{3}})=a^2-b$,

and $(a^{\frac{1}{2}}x^{-\frac{1}{3}}+a^{-\frac{1}{2}}x^{\frac{1}{3}}+a^{\frac{1}{2}}y^{-\frac{1}{4}}+a^{-\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{3}}y^{-\frac{1}{4}}+x^{-\frac{1}{3}}y^{\frac{1}{4}}+3)$

$\div(a^{-\frac{1}{2}}+x^{-\frac{1}{3}}+y^{-\frac{1}{4}})=a^{\frac{1}{2}}+x^{\frac{1}{3}}+y^{\frac{1}{4}}$.

68. The square of $\sqrt[6]{ax^4}=\sqrt[3]{ax^4}$; the cube $=x^2\sqrt{a}$; the fourth power $=x^2\sqrt[3]{a^2x^2}$; &c.

69. The square of $\frac{1}{a}\sqrt[3]{2ax}=\sqrt[3]{\frac{4x^2}{a^4}}$, the fifth power $=\frac{2x}{a^3}\sqrt[3]{\frac{4x^2}{a}}$ and the sixth power $=\frac{4x^2}{a^4}$.

70. The square of $(a+x)\sqrt[4]{a^2-x^2}=(a+x)^{\frac{5}{2}}\sqrt{a-x}$,

and the cube of $(a-x)(ax-x^2)^{\frac{1}{2}}=x^{\frac{3}{2}}(a-x)^{\frac{9}{2}}$.

71. The square of $2\sqrt{a}+3\sqrt{x}=4a+9x+12\sqrt{ax}$, and the cube of $\sqrt[3]{x}-\sqrt{a}=x-3a^{\frac{1}{2}}x^{\frac{2}{3}}+3ax^{\frac{1}{3}}-a\sqrt{a}$.

72. The squares of $a+\sqrt{x^2-a^2}$ and $\sqrt{1+x}-\sqrt{1-x}$ are $x^2+2a\sqrt{x^2-a^2}$ and $2-2\sqrt{1-x^2}$.

73. Of $\sqrt{1-x+x^2}+\sqrt{1+x-x^2}$ and $\sqrt{a-\sqrt{a^2-x}}\pm\sqrt{a+\sqrt{a^2-x}}$ are $2+2\sqrt{1-x^2+2x^3-x^4}$ and $2(a\pm\sqrt{x})$.

74. Of $a+2\sqrt{ab}+b=a^2+6ab+b^2+4(a+b)\sqrt{ab}$,

and of $a^{\frac{1}{2}}x^{-\frac{1}{2}}+a^{\frac{1}{4}}x^{-\frac{1}{4}}=\frac{a}{x}+2\sqrt[4]{\left(\frac{a}{x}\right)^3}+\sqrt{\frac{a}{x}}$.

75. The cube of $\sqrt[3]{x}-\sqrt[3]{y}=x-y-3\sqrt[3]{xy}(\sqrt[3]{x}-\sqrt[3]{y})$

and of $(a+x)^{\frac{1}{3}}+(a+x)^{-\frac{1}{3}}=\frac{(a+x)^2+1}{a+x}+3\left\{\frac{(a+x)^{\frac{2}{3}}+1}{(a+x)^{\frac{1}{3}}}\right\}$.

76. The cube of $ax^{\frac{1}{2}}-by^{\frac{1}{2}}=a^3x^{\frac{3}{2}}-3a^2bxy^{\frac{1}{2}}+3ab^2x^{\frac{1}{2}}y-b^3y^{\frac{3}{2}}$,

and the 4th power of $a^{\frac{1}{2}}+b^{\frac{1}{4}}=a^2+4a^{\frac{3}{2}}b^{\frac{1}{4}}+6ab^{\frac{1}{2}}+4a^{\frac{1}{2}}b^{\frac{3}{4}}+b$.

77. The cube of $\sqrt[3]{\frac{x^2}{a}}-\sqrt[3]{\frac{a^2}{x}}=\frac{x^3-a^3}{ax}-3(x-a)$,

the 4th power of $\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}=\frac{a^2}{x^2}+\frac{x^2}{a^2}+4\left(\frac{a}{x}+\frac{x}{a}\right)+6$.

78. The square roots of $a^2b\sqrt{x}$, $x\sqrt[3]{a^2b^2}$ and $(a+b)\sqrt[5]{a^3+a^2b}$, are $a\sqrt[4]{b^2x}$, $\sqrt{x}\sqrt[3]{ab}$ and $\sqrt[5]{a(a+b)^3}$.

79. The cube roots of $a\sqrt{a}$, $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$ and $\frac{\sqrt[4]{(ax-x^2)^5}}{x^3}$

are $\sqrt{a}$, $\left(\frac{2a}{3}\right)^{\frac{1}{9}}$ and $\sqrt[12]{\frac{(a-x)^5}{x^7}}$.

80. The fourth root of

$$\frac{\sqrt{ax+x^2}\sqrt[3]{(a^2-x^2)^4}}{x} = \frac{(a-x)^{\frac{1}{3}}(a+x)^{\frac{11}{24}}}{x^{\frac{1}{8}}}$$

and the fifth root of

$$\frac{\sqrt{3ax-x^2}\sqrt[3]{2ay-y^2}}{\sqrt[4]{x^2y+xy^2}} = \sqrt[60]{(3a-x)^6(2a-y)^4\frac{x^3y}{(x+y)^3}}.$$

81. The square roots of $a^2-4a\sqrt{b}+4b$ and $ax^2+by^2-2xy\sqrt{ab}$, are $a-2\sqrt{b}$ and $x\sqrt{a}-y\sqrt{b}$.

82. Of $4a^2-12a\sqrt{b}+9b+12-18a^{-1}\sqrt{b}+9a^{-2}$ and $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)\sqrt{2}+\frac{5}{2}$, are $2a-3\sqrt{b}+3a^{-1}$ and $\frac{x}{y}-\frac{1}{\sqrt{2}}+\frac{y}{x}$.

83. Of $\frac{4x^{\frac{3}{2}}}{9a^{\frac{1}{2}}}-\frac{4x^{\frac{3}{4}}y^{\frac{1}{4}}}{5a^{\frac{1}{4}}b^{\frac{3}{4}}}+\frac{9y^{\frac{1}{2}}}{25b^{\frac{3}{2}}}$ is $\frac{2x^{\frac{3}{4}}}{3a^{\frac{1}{4}}}-\frac{3y^{\frac{1}{4}}}{5b^{\frac{3}{4}}}$ and of

$16x^{\frac{10}{3}}+\frac{16x^{\frac{15}{6}}}{y^{\frac{2}{3}}}+\frac{28x^{\frac{5}{3}}}{y^{\frac{4}{3}}}+\frac{12x^{\frac{5}{6}}}{y^2}+\frac{9}{y^{\frac{8}{3}}}$ is $4x^{\frac{5}{3}}+\frac{2x^{\frac{5}{6}}}{y^{\frac{2}{3}}}+\frac{3}{y^{\frac{4}{3}}}$.

84. Of $1+\frac{41a}{16}-\frac{3+3a}{2}\sqrt{a}+a^2$ and $4a^2+bx-y^2\pm 2\sqrt{4a^2bx-bxy^2}$ are $1-\frac{3}{4}\sqrt{a}+a$ and $\sqrt{4a^2-y^2}\pm\sqrt{bx}$.

85. Of $a+b+c+2\sqrt{ac+bc}$ and $a+b+c+2\left\{\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\right\}$ are $\sqrt{a+b}+\sqrt{c}$ and $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

86. Of $a+b+c+d+2\sqrt{ad+bd+cd}$ and $a+b+c+d+2\sqrt{ac+bc+ad+bd}$ are $\sqrt{a+b+c}+\sqrt{d}$ and $\sqrt{a+b}+\sqrt{c+d}$.

87. The cube roots of $ax - 3\sqrt[3]{ax^4} + 3\sqrt[3]{a^2x^8} - x^5$ and $\sqrt{a^3} + 3a - 5 + \frac{3}{a} - \sqrt{\frac{1}{a^3}}$, are $\sqrt[3]{ax} - x$ and $\sqrt{a} + 1 - \frac{1}{\sqrt{a}}$.

88. The fourth roots of $a^2 + b^2 - 4\sqrt{ab}\,(a+b) + 6ab$ and $2x^4 - 4x^3\sqrt[4]{8} + 6x^2\sqrt{2} - 4x\sqrt[4]{2} + 1$, are $\sqrt{a} - \sqrt{b}$ and $x\sqrt[4]{2} - 1$.

89. The square roots of $2a \pm 2\sqrt{a^2 - x^2}$ and $ax \pm 2a\sqrt{ax - a^2}$ are $\sqrt{a+x} \pm \sqrt{a-x}$ and $\sqrt{ax - a^2} \pm a$.

90. Of $\frac{a^2}{4} + \frac{x}{2}\sqrt{a^2 - x^2}$ and $\frac{3a}{x} + \sqrt{\frac{12a^3b^2}{c^2x} - \frac{4a^4b^4}{c^4}}$ are $\frac{x}{2} + \frac{\sqrt{a^2 - x^2}}{2}$ and $\frac{ab}{c} + \sqrt{\frac{3a}{x} - \frac{a^2b^2}{c^2}}$.

91. Of $a^2 - ax + \frac{x^2}{4} + 2\sqrt{a^3x - 2a^2x^2 + \frac{ax^3}{4}}$.

and $\sqrt[m]{a^{-\frac{2}{n}}} + \sqrt[r]{a^{\frac{1}{n}}b} + 2\sqrt[2r]{a^{\frac{m-2r}{mn}}b}$,

are $\sqrt{a^2 - 2ax + \frac{x^2}{4}} + \sqrt{ax}$ and $a^{-\frac{1}{mn}} + a^{\frac{1}{2rn}}b^{\frac{1}{2r}}$.

92. Of $2a^2 + 2\sqrt{a^4 - x^4}$ and $a + x + \sqrt{2ax + x^2}$, are $\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2}$ and $\sqrt{\frac{2a + x}{2}} + \sqrt{\frac{x}{2}}$.

93. Of $3 \pm 2\sqrt{2}$ and $4 \pm 2\sqrt{3}$, are $\sqrt{2} \pm 1$ and $\sqrt{3} \pm 1$.

94. Of $7 \pm 4\sqrt{3}$ and $11 \pm 6\sqrt{2}$, are $2 \pm \sqrt{3}$ and $3 \pm \sqrt{2}$.

95. Of $32 \pm 10\sqrt{7}$, $28 \pm 5\sqrt{12}$ and $36 \pm 10\sqrt{11}$, are $5 \pm \sqrt{7}$, $5 \pm \sqrt{3}$ and $5 \pm \sqrt{11}$.

96. Of $5 \pm \sqrt{24}$, $87 \pm 12\sqrt{42}$ and $12 \pm 2\sqrt{35}$,

are $\sqrt{3} \pm \sqrt{2}$, $3\sqrt{7} \pm 2\sqrt{6}$ and $\sqrt{5} \pm \sqrt{7}$.

97. Of $2 \pm \sqrt{3}$, $4\frac{1}{2} \pm 2\sqrt{2}$ and $3\frac{1}{2} \pm \sqrt{10}$,

are $\frac{1}{2}\{\sqrt{6} \pm \sqrt{2}\}$, $\frac{1}{2}\{4 \pm \sqrt{2}\}$ and $\frac{1}{2}\{2 \pm \sqrt{10}\}$.

98. Of $\sqrt{18} \pm 4$, $\sqrt{27} \pm 2\sqrt{6}$ and $4\sqrt{3} \pm 6$,

are $\sqrt[4]{8} \pm \sqrt[4]{2}$, $\sqrt[4]{12} \pm \sqrt[4]{3}$ and $\sqrt[4]{27} \pm \sqrt[4]{3}$.

99. Of $3\sqrt{5} \pm 2\sqrt{10}$, $3\sqrt{6} \pm 4\sqrt{3}$ and $5\sqrt{2} \pm 2\sqrt{8}$,

are $\sqrt[4]{20} \pm \sqrt[4]{5}$, $\sqrt[4]{24} \pm \sqrt[4]{6}$ and $\sqrt[4]{32} \pm \sqrt[4]{2}$.

100. Of $14 + 4\sqrt{2} - 2\sqrt{5} + 4\sqrt{10}$

and $9 - 2\sqrt{3} + 2\sqrt{5} - 2\sqrt{15}$,

are $1 + 2\sqrt{2} - \sqrt{5}$ and $1 - \sqrt{3} + \sqrt{5}$.

101. Of $14 + \sqrt{32} - \sqrt{48} + \sqrt{80} - \sqrt{24} + \sqrt{40} - \sqrt{60}$

and $11 - 2\sqrt{2} + 2\sqrt{3} - 2\sqrt{5} - 2\sqrt{6} + 2\sqrt{10} - 2\sqrt{15}$,

are $2 + \sqrt{2} - \sqrt{3} + \sqrt{5}$ and $1 - \sqrt{2} + \sqrt{3} - \sqrt{5}$.

102. The cube roots of $10 \pm 6\sqrt{3}$ and $38 \pm 17\sqrt{5}$, are $1 \pm \sqrt{3}$ and $2 \pm \sqrt{5}$.

103. The fourth roots of $17 \pm 12\sqrt{2}$ and $14 \pm 8\sqrt{3}$, are $1 \pm \sqrt{2}$ and $\dfrac{\sqrt{3} \pm 1}{\sqrt[4]{2}}$.

104. Prove the truth of the following expressions:

$$\sqrt{1 \pm x} = 1 \pm \frac{x}{2} - \frac{x^2}{8} \pm \frac{x^3}{16} - \frac{5x^4}{128} \pm \&c.$$

$$\sqrt{a^2 \pm x^2} = a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} \pm \&c.$$

$$\sqrt[3]{1 \pm x} = 1 \pm \frac{x}{3} - \frac{x^2}{9} \pm \frac{5x^3}{81} - \frac{10x^4}{243} \pm \&c.$$

$$\sqrt[3]{a^3 \pm x^3} = a \pm \frac{x^3}{3a^2} - \frac{x^6}{9a^5} \pm \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} \pm \&c.$$

105. $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}} = \left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}}$, $\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}} = a^{-\frac{mp}{nq}} = \left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}}$.

106. $$\left\{\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}\right\}^{\frac{r}{s}} = a^{\frac{mpr}{nqs}} = \left\{\left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}}\right\}^{-\frac{r}{s}}$$

$= \left\{\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}}\right\}^{\frac{r}{s}} = \left\{\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}}\right\}^{-\frac{r}{s}}$ and $\left\{\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}\right\}^{-\frac{r}{s}} = a^{-\frac{mpr}{nqs}}$

$$= \left\{\left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}}\right\}^{\frac{r}{s}} = \left\{\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}}\right\}^{\frac{r}{s}} = \left\{\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}}\right\}^{-\frac{r}{s}}.$$

107. $(x^{2m} + x^{2n})^{\frac{1}{mn}} = x^{\frac{1}{m}+\frac{1}{n}} \left\{x^{m-n} + x^{n-m}\right\}^{\frac{1}{mn}}$ and

$$\frac{a^2-b^2}{a^2+b^2}\left\{x^{\frac{1}{p}} + x^{\frac{1}{q}}\right\} - 2x^{\frac{p+q}{2pq}} = \frac{a^2-b^2}{a^2+b^2}\left\{x^{\frac{1}{2p}} - x^{\frac{1}{2q}}\right\}^2 - \frac{4b^2}{a^2+b^2}x^{\frac{p+q}{2pq}}.$$

108. $\sqrt{\dfrac{1+x}{1-x}} - \sqrt{\dfrac{1-x}{1+x}} = x\left\{\sqrt{\dfrac{1+x}{1-x}} + \sqrt{\dfrac{1-x}{1+x}}\right\}$,

and $(1+x)\left\{\sqrt{1+2x} - \dfrac{1}{\sqrt{1+2x}}\right\} = x\left\{\sqrt{1+2x} + \dfrac{1}{\sqrt{1+2x}}\right\}$.

109. $$(1+x+x^2)\left\{\frac{1+x}{\sqrt{1+x^2}} - \frac{\sqrt{1+x^2}}{1+x}\right\}$$

$= x\left\{\dfrac{1+x}{\sqrt{1+x^2}} + \dfrac{\sqrt{1+x^2}}{1+x}\right\}$ and $(1-x)^{\frac{3}{2}}\sqrt{1+x}$

$$\times \left\{\tfrac{1}{2}\sqrt{\frac{1-x}{1+x}} + \tfrac{1}{2}\sqrt{\frac{1+x}{1-x}}\right\} = 1 - x.$$

110. $\sqrt{2x} - \sqrt{3a}$ and $\sqrt[3]{ax} + \sqrt[3]{by}$ are equal to $\sqrt{2x+3a-2\sqrt{6ax}}$ and $\sqrt[3]{ax+by+3\sqrt[3]{a^2bx^2y}+3\sqrt[3]{ab^2xy^2}}$.

111. $(\sqrt{x}-1)\sqrt{x+\sqrt{x}} = \sqrt{(x-1)(x-\sqrt{x})}$

and $\sqrt{y^2-2} + \dfrac{1}{\sqrt{y^2-2}} = \sqrt{\dfrac{y^4-2y^2+1}{y^2-2}}$.

112. $\sqrt[3]{3}+\sqrt[3]{5}$ and $\sqrt[4]{4}-\sqrt[4]{2}$, are equivalent to $\sqrt[3]{8+3\sqrt[3]{45}+3\sqrt[3]{75}}$ and $\sqrt[4]{6+12\sqrt{2}-8\sqrt[4]{8}-8\sqrt[4]{2}}$.

113. If $\sqrt{(a+x)^2+b^2}+\sqrt{(a-x)^2+b^2}=2c$, prove that $(a^2-c^2)(x^2-c^2)=b^2c^2$; and if $\sqrt{(a+x)^2+y^2} \times \sqrt{(a-x)^2+y^2}=a^2$, prove that $(x^2+y^2)^2=2a^2(x^2-y^2)$.

114. $\dfrac{1}{x+\sqrt{x^2-1}}$ and $\dfrac{a^2}{a+x-\sqrt{2ax+x^2}}$, are equivalent to $x-\sqrt{x^2-1}$ and $a+x+\sqrt{2ax+x^2}$.

115. $\dfrac{2\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}$ and $\dfrac{2ax}{\sqrt[3]{7}-\sqrt[3]{5}}$, are equivalent to

$$\{\sqrt[4]{125}-\sqrt[4]{75}+\sqrt[4]{45}-\sqrt[4]{27}\}\sqrt{ab}$$

and $\{\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}\}\,2ax$.

116. $\dfrac{4x}{1+\sqrt{2}+\sqrt{3}}$ and $\dfrac{12y}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$, are equivalent to $(2+\sqrt{2}-\sqrt{6})\,x$ and $(2\sqrt{3}+3\sqrt{2}+\sqrt{30})\,y$.

117. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ and $\dfrac{\sqrt[3]{a}}{\sqrt[3]{x}+\sqrt[3]{y}}$, are equivalent to

$$\frac{a+\sqrt{ab}}{a-b} \text{ and } \frac{\sqrt[3]{ax^2}-\sqrt[3]{axy}+\sqrt[3]{ay^2}}{x+y}.$$

118. $\dfrac{1}{\sqrt[m]{a+\sqrt{b}}}$ and $\sqrt[m]{\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}}$, are equivalent to $\sqrt[m]{\dfrac{a-\sqrt{b}}{a^2-b}}$ and $\sqrt[m]{\dfrac{a+b+2\sqrt{ab}}{a-b}}$.

119. $\dfrac{a^{\frac{3}{2}}-a}{a^2+a^{\frac{1}{2}}}=\dfrac{a-a^{\frac{1}{2}}}{a^{\frac{3}{2}}+1}$, $\dfrac{a-x}{a-2\sqrt{ax}+x}=\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$,

and $\dfrac{\sqrt[3]{c^4}+\sqrt[3]{c^2d^2}-\sqrt[3]{c^3d}}{\sqrt[3]{c^4}+\sqrt[3]{cd^3}-\sqrt[3]{c^3d}-\sqrt[3]{d^4}}=\dfrac{c^{\frac{2}{3}}}{c^{\frac{2}{3}}-d^{\frac{2}{3}}}$.

120. $\dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}=2\sqrt[3]{5}+\sqrt[3]{20}+\sqrt[3]{10}$, and $\dfrac{\sqrt{10}}{\sqrt{2}-\sqrt[3]{3}}$

$=-\sqrt{5}\,(8+6\sqrt[3]{3}+4\sqrt[3]{9})-\sqrt{10}\,(6+4\sqrt[3]{3}+3\sqrt[3]{9})$.

121. $\sqrt{-ab}=\sqrt{ab}\sqrt{-1}=\sqrt{a}\sqrt{b}\sqrt{-1}$

$=\sqrt{-a}\sqrt{b}=\sqrt{a}\sqrt{-b}$, and $\sqrt{-\dfrac{a}{b}}=\sqrt{\dfrac{a}{b}}\sqrt{-1}$

$$=\frac{\sqrt{a}}{\sqrt{b}}\sqrt{-1}=\frac{\sqrt{-a}}{\sqrt{b}}=-\frac{\sqrt{a}}{\sqrt{-b}}.$$

122. $a+\sqrt{-b^2}+c\sqrt{-1}-\sqrt{-d^2}$

$=a+(b+c-d)\sqrt{-1}$, and $(x-a)\sqrt{-1}+bc\sqrt{-\dfrac{1}{a^2}}$

$-k\sqrt{-y^2}=(x-a+\dfrac{bc}{a}-ky)\sqrt{-1}$.

123. $2\sqrt{-4}+3\sqrt{-9}+5\sqrt{-16}=33\sqrt{-1}$,

and $4\sqrt{-8}+3\sqrt{-32}-19\sqrt{-2}+6\sqrt{-3}-4\sqrt{-27}$

$$=(\sqrt{2}-6\sqrt{3})\sqrt{-1}.$$

124. $\sqrt{-ab}\times\sqrt{-bc}=-b\sqrt{ac}$,

and $a\sqrt{-x}\times b\sqrt{-y}=-ab\sqrt{xy}$.

125. $(a+\sqrt{-b})\times(a-\sqrt{-b})=a^2+b$,

and $(a+\sqrt{-a^2})\times(a-\sqrt{-a^2})=2a^2$.

126. $(\sqrt{a^2-b^2}+b\sqrt{-1})\times(\sqrt{a^2-b^2}-b\sqrt{-1})=a^2$,

and $(\sqrt{a^2+b^2}+c\sqrt{-1})\times(\sqrt{a^2+b^2}-c\sqrt{-1})=a^2+b^2+c^2$.

127. $(2-\sqrt{-5})\times(3+\sqrt{-5})=11-\sqrt{-5}$,

and $(\sqrt{2}-3\sqrt{-5})\times(\sqrt{7}-\sqrt{-3})=\sqrt{14}-3\sqrt{15}$
$-(3\sqrt{35}+\sqrt{6})\sqrt{-1}$.

128. $(2x+1-\sqrt{-3})\times(2x+1+\sqrt{-3})\times(x-1)=4(x^3-1)$,

and $(x-1-\sqrt{-2})\times(x-1+\sqrt{-2})\times(x-2+\sqrt{-3})\times$
$(x-2-\sqrt{-3})=x^4-6x^3+18x^2-26x+21$.

129. $(x+a)\times(x-a)\times\left\{x-\frac{a}{2}(1+\sqrt{-3})\right\}$

$$\times\left\{x-\frac{a}{2}(1-\sqrt{-3})\right\}\times\left\{x+\frac{a}{2}(1+\sqrt{-3})\right\}$$

$$\times\left\{x+\frac{a}{2}(1-\sqrt{-3})\right\}=x^6-a^6,$$

and $(x-a\sqrt{-1})\times(x+a\sqrt{-1})\times$

$$\left\{x+\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}\times\left\{x+\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}$$

$$\times\left\{x-\frac{a}{2}(\sqrt{3}+\sqrt{-1})\right\}\times\left\{x-\frac{a}{2}(\sqrt{3}-\sqrt{-1})\right\}=x^6+a^6.$$

130. $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=2\left(\dfrac{a^2-b^2}{a^2+b^2}\right)$,

and $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}-\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\dfrac{4ab\sqrt{-1}}{a^2+b^2}$.

131. $\dfrac{a+b\sqrt{-1}}{x+y\sqrt{-1}}+\dfrac{a-b\sqrt{-1}}{x-y\sqrt{-1}}=\dfrac{2(ax+by)}{x^2+y^2}$,

and $\dfrac{a+b\sqrt{-1}}{x+y\sqrt{-1}}\times\dfrac{a-b\sqrt{-1}}{x-y\sqrt{-1}}=\dfrac{a^2+b^2}{x^2+y^2}$.

132. $a\sqrt{-1}\div\sqrt{-a^2b}=\dfrac{1}{\sqrt{b}}$, and

$\left\{ab+xy-(ay-bx)\sqrt{-1}\right\}\div(a+x\sqrt{-1})=b-y\sqrt{-1}$.

133. $(\sqrt{-12}+\sqrt{-6}+\sqrt{-9})\div\sqrt{-3}=2+\sqrt{2}+\sqrt{3}$,
and

$$\{14-\sqrt{15}-(7\sqrt{3}+2\sqrt{5})\sqrt{-1}\}\div(7-\sqrt{-5})=2-\sqrt{-3}.$$

134. $1\div\sqrt{-1}=-\sqrt{-1},\ -a^2\div a\sqrt{-1}=a\sqrt{-1}$,
and $\sqrt{-1}\sqrt{x^2-1}\div\sqrt{1-x}=\sqrt{1+x}$.

135. $(a\pm\sqrt{-x^2})^2=a^2-x^2\pm 2ax\sqrt{-1}$,
and $(a\pm\sqrt{-x^2})^3=a^3-3ax^2\pm(3a^2x-x^3)\sqrt{-1}$.

136. $(x+\sqrt{-1}\sqrt{1-x^2})^2=2x^2-1+2x\sqrt{x^2-1}$,
and $(3a^2-2a\sqrt{-1}+\sqrt{-2})^2$
$$=9a^4-12a^3\sqrt{-1}-2a^2(2-3\sqrt{-2})+4a\sqrt{2}-2.$$

137. $\sqrt{a^2-b^2\pm 2ab\sqrt{-1}}=a\pm b\sqrt{-1}$,
and $\sqrt[3]{1+3a\sqrt{-1}-3a^2-a^3\sqrt{-1}}=1+a\sqrt{-1}$.

138. $\sqrt{4ab+2(a^2-b^2)\sqrt{-1}}=a+b+(a-b)\sqrt{-1}$,
and $\sqrt{\dfrac{2c^2}{d^2}\sqrt{-1}}=\dfrac{c}{d}(1+\sqrt{-1})$.

139. $\sqrt{a^4x^4-a^3b^2-a^2b^3-2a^3bx^2\sqrt{-(a+b)}}$
$=a^2x^2-ab\sqrt{-(a+b)}$, and

$$\sqrt{\frac{25a^2d}{c^2}-\frac{4a^2b}{d}-\frac{20a^2}{c}\sqrt{-b}}=\frac{5a}{c}\sqrt{d}-2a\sqrt{-\frac{b}{d}}.$$

140. $\sqrt{31\pm 12\sqrt{-5}}=6\pm\sqrt{-5}$,
and $\sqrt{24\sqrt{-1}-7}=3+4\sqrt{-1}$.

141. $\sqrt[3]{-5+\sqrt{-2}}=1+\sqrt{-2}$,
and $\sqrt[4]{-28+16\sqrt{-2}}=2+\sqrt{-2}$.

142. $\sqrt{a+b\sqrt{-1}}+\sqrt{a-b\sqrt{-1}}$

$=\sqrt{2a+2\sqrt{a^2+b^2}}$, and $\sqrt{x^2+b^2}\pm\sqrt{-1}\sqrt{x^2-a^2}$

$=\sqrt{a^2+b^2\pm 2\sqrt{a^2b^2+(a^2-b^2)x^2-x^4}}$.

143. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}=\sqrt{-1}$, $\dfrac{a-a\sqrt{-1}}{b+b\sqrt{-1}}=-\dfrac{a}{b}\sqrt{-1}$,

and $\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}=\dfrac{a^2-b^2-2ab\sqrt{-1}}{a^2+b^2}$.

144. $\dfrac{5-\sqrt{-2}}{1+\sqrt{-2}}=1-2\sqrt{-2}$, $\dfrac{1}{1+2\sqrt{-1}}=\dfrac{1}{5}(1-2\sqrt{-1})$,

and $\dfrac{21}{3-2\sqrt{-3}}=3+2\sqrt{-3}$.

145. If $(a+b\sqrt{-1})^{\frac{1}{5}}=x+y\sqrt{-1}$, it is required to prove that $(a-b\sqrt{-1})^{\frac{1}{5}}=x-y\sqrt{-1}$.

146. Reduce $\sqrt[4]{-1}$, $1\pm\sqrt[4]{-4}$ and $3\sqrt[4]{-1}+5\sqrt[8]{-1}$ to the form $\alpha\pm\beta\sqrt{-1}$.

## CHAP. VI.

### I. SIMPLE EQUATIONS.

1. In $7x-3=5x+13$, $x=8$.
2. In $3x+5=10x-16$, $x=3$.
3. In $15x-24=20+\dfrac{x}{3}$, $x=3$.
4. In $x-\dfrac{x}{2}+\dfrac{x}{3}-\dfrac{x}{4}=7$, $x=12$.
5. In $\dfrac{x}{2}-\dfrac{5x+4}{3}=\dfrac{4x-9}{3}$, $x=\dfrac{2}{3}$.

6. In $\frac{x+1}{2}+\frac{x+2}{3}=14+\frac{5-x}{4}$, $x=13$.

7. In $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}=16$, $x=13$.

8. In $\frac{x+6}{4}+\frac{16-3x}{12}=\frac{x+9}{6}$, $x=8$.

9. In $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{5x+1}{4}$, $x=7$.

10. In $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}$, $x=18$.

11. In $\frac{x-7}{11}-\frac{3x-5}{7}+\frac{125}{77}=2x-17$, $x=8$.

12. In $\frac{5x-7}{3}-\frac{3x-2}{7}=\frac{x-5}{4}$, $x=\frac{67}{83}$.

13. In $\frac{x+3}{2}-\frac{11-x}{5}=\frac{3x-1}{20}+3\frac{1}{5}$, $x=7$.

14. In $x-\frac{x-2}{3}=5\frac{3}{4}-\frac{x+10}{5}+\frac{x}{4}$, $x=5$.

15. In $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36$, $x=9$.

16. In $\frac{3x+7}{14}-\frac{2x-7}{21}+2\frac{3}{4}=\frac{x-4}{4}$, $x=35$.

17. In $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, $x=72$.

18. In $\frac{4x-21}{7}+7\frac{3}{4}+\frac{7x-28}{3}=x+3\frac{3}{4}-\frac{9-7x}{8}$, $x=7$.

19. In $\frac{x-1\frac{25}{26}}{2}-\frac{2-6x}{13}=x-\frac{5x-\frac{10-3x}{4}}{39}$, $x=11$.

20. In $\dfrac{\dfrac{3x}{4}-2}{8}+19=x-\dfrac{\dfrac{x}{2}+1-\frac{1}{5}(x-4)+x}{11}$, $x=24$.

21. In $23+\dfrac{5x-1}{11}+\dfrac{3x-2}{5}-\dfrac{11x-3}{12}=\dfrac{13x-15}{3}-\dfrac{8x-2}{7}$, $x=9$.

22. In $\dfrac{17-3x}{5}-\dfrac{2x+1}{1\frac{1}{2}}=\dfrac{29-11x}{3}$, $x=4$.

23. In $\dfrac{3x-1}{4}-\dfrac{5x+1}{6}+\dfrac{2x-11}{9}-\dfrac{7x-13}{12}=\dfrac{11x+7}{18}-\dfrac{499-13x}{36}-14$, $x=19$.

24. In $25+\dfrac{15x-23}{29}-\dfrac{261-11x}{37}+\dfrac{137-3x}{43}=\dfrac{95x-22}{59}-\dfrac{6x-504}{67}$, $x=17$.

25. In $.15x+.2-.875x+.375=.0625x-1$, $x=2$.

26. In $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$, $x=\dfrac{1}{2}$.

27. In $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}$, $x=20$.

28. In $\dfrac{2x+8\frac{1}{2}}{9}-\dfrac{13x-2}{17x-32}+\dfrac{x}{3}=\dfrac{7x}{12}-\dfrac{x+16}{36}$, $x=4$.

29. In $\dfrac{41-35x}{105}-\dfrac{7-2x^2}{14(x-1)}=\dfrac{1+3x}{21}-\dfrac{2x-\frac{11}{5}}{6}$, $x=4$.

30. In $\dfrac{6x-7\frac{1}{3}}{13-2x}+2x+\dfrac{1+16x}{24}=4\tfrac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}$, $x=1\frac{1}{2}$.

31. In $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=5+\dfrac{23}{x+1}$, $x=3\frac{3}{8}$.

32. In $\dfrac{3-2x}{1-2x}-\dfrac{5-2x}{7-2x}=1-\dfrac{4x^2-2}{7-16x+4x^2}$, $x=-\dfrac{7}{8}$.

33. In $\sqrt{x+9}=1+\sqrt{x}$, $x=16$.

34. In $(\sqrt{x}+28)(\sqrt{x}+6)=(\sqrt{x}+38)(\sqrt{x}+4)$, $x=4$.

35. In $\sqrt{x}-4=\dfrac{259-10x}{\sqrt{x}+4}$, $x=25$.

36. In $x+\sqrt{2ax+x^2}=a$, $x=\dfrac{a}{4}$.

37. In $\sqrt{4+\sqrt{x^4-x^2}}=x-2$, $x=2\frac{1}{8}$.

38. In $ax+b^2=a^2+bx$, $x=a+b$.

39. In $bx+2x-a=3x+2c$, $x=\dfrac{a+2c}{b-1}$.

40. In $\dfrac{5(x-a)}{6}-\dfrac{2x-3b}{5}=10a+11b$, $x=25a+24b$.

41. In $\dfrac{3x-a}{b}+\dfrac{x+2b}{c}=\dfrac{7x}{c}-\dfrac{a}{4}$, $x=\dfrac{8b^2-4ac+abc}{12(2b-c)}$.

42. In $\dfrac{bx}{a}-\dfrac{d}{c}=\dfrac{a}{b}-\dfrac{cx}{d}$, $x=\dfrac{ad}{bc}$.

43. In $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2$, $x=\dfrac{ac}{b}$.

44. In $\sqrt[m]{a+x}=\sqrt[2m]{x^2+5ax+b^2}$, $x=\dfrac{a^2-b^2}{3a}$.

45. In $\sqrt[3]{ax+b}=\sqrt[3]{cx+d}$, $x=\dfrac{d-b}{a-c}$.

46. In $\dfrac{ax^2+a^3}{a+x}=ax+b^2$, $x=a\left\{\dfrac{a^2-b^2}{a^2+b^2}\right\}$.

47. In $\sqrt{a+x}-\sqrt{\dfrac{a}{a+x}}=\sqrt{2a+x}$,

$$x=\left(\frac{1-2\sqrt{a}-a}{2+\sqrt{a}}\right)\sqrt{a}.$$

48. In $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\sqrt{b}$, $x=\dfrac{2a\sqrt{b}}{1+b}$.

49. In $\dfrac{\sqrt{a+\sqrt{x}}}{\sqrt[4]{x}}+\dfrac{\sqrt{a-\sqrt{x}}}{\sqrt[4]{x}}=\sqrt[4]{x},\ x=4(a-1).$

50. In $\sqrt{\dfrac{a^2}{x}+b}-\sqrt{\dfrac{a^2}{x}-b}=\sqrt{c},\ x=\dfrac{4a^2c}{4b^2+c^2}.$

51. In $\sqrt{a+x}=\sqrt{b}+\sqrt{x},\ x=\dfrac{(a-b)^2}{4b}.$

52. In $\sqrt{x}+\sqrt{a+x}=\dfrac{na}{\sqrt{a+x}},\ x=\dfrac{(n-1)^2}{2n-1}a.$

53. In $\sqrt{x+a}+\sqrt{x-a}=\dfrac{b}{\sqrt{x+a}},\ x=\dfrac{2a^2-2ab+b^2}{2(b-a)}.$

54. In $\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{a}}=\sqrt{\dfrac{1}{a}+\sqrt{\dfrac{4}{ax}+\dfrac{9}{x^2}}},\ x=4a.$

55. In $\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}}=\sqrt[3]{b},$

$$x=\frac{8a^3+15a^2b+6ab^2-b^3}{27b}.$$

56. In $\frac{1}{2}\sqrt{\sqrt{x}+3a^2}-\frac{1}{2}\sqrt{\sqrt{x}-3a^2}=\sqrt[4]{\dfrac{a^2x}{b^2}},$

$$x=\frac{9a^3b^2}{4(b-a)}.$$

## Problems.

1. What number is that to which if its third and fourth parts be added, the sum will exceed its sixth part by 17?

The required number is 12.

2. What number is that from which if 50 be subtracted, the remainder will be equal to its half together with its fourth and sixth parts? The number required is 600.

3. Find a number which when multiplied by 4 becomes as much above 30 as it is now below it. The number is 12.

4. Two persons at a distance of 240 leagues, set out to meet each other, and travel at the rates of 7 and 8 leagues a day respectively: when and where will they meet?

They will meet at the end of 16 days, having travelled 112 and 128 leagues respectively.

5. A labourer was engaged for 36 days upon the condition that for every day he worked he was to receive half a crown, and for every day he was absent, to forfeit eighteenpence: and at the end of his time he received £2. 18*s.*: how many days did he work, and how many was he absent?

He worked 28 days and was absent 8 days.

6. *A* is three times as rich as *B*, and if *B* give him £50, *A* becomes four times as rich as *B*: required the property of each. *A*'s property is £750. and *B*'s is £250.

7. *A* possesses £600. and *B* £480; what sum must *A* receive of *B* that he may become twice as rich as *B*?

He must receive £120.

8. *A*'s money exceeds *B*'s and *C*'s by £240. and £320. respectively, and that of *B* and *C* together is £600: required the sum possessed by each.

*A* has £580, *B* has £340, and *C* has £260.

9. *A*, *B* and *C* together possess £600; *A*, *B* and *D* together £720; *A*, *C* and *D* together £900; and *B*, *C* and *D* together £1020; what is the sum possessed by each?

*A* has £60, *B* has £180, *C* has £360, and *D* has £480.

10. *A* and *B* together possess £150, and *C* has £50. more than *D*: also *A* has twice as much as *C*, and *B* thrice as much as *D*: required the money of each.

*A* has £120, *B* has £30, *C* has £60, and *D* has £10.

11. *A* merchant after allowing £1600. for his annual expenditure, increases his property every year by a fourth part, and at the end of two years is £9~~6~~900. richer than at first: what property does he begin with?

His original property is £24000.

11. Let $4x$ = orig.l property

$\therefore 5x - 1600$ = capital at end of 1st year

$\therefore \frac{5}{4}(5x - 1600) - 1600 = 4x + 9900$

$5x - 1600 - 1280 = \frac{16x}{5} + 7920 \quad \therefore 9x = 54000$

$4x = 24000$

12. From a sum of money is first taken away £20. more than its half: from the remainder £30. more than its third part, and from what then remained £40. more than its fourth part, and afterwards nothing remains: what is the sum? The sum is £290.

13. Find two magnitudes whose sum is $s$ and difference $d$.

$$\text{The greater is } \frac{s+d}{2} \text{ and the less is } \frac{s-d}{2}.$$

14. $A$ has $m$ times as much money as $B$: also, if they receive £$a$. and £$b$. respectively, $A$ will have $n$ times as much as $B$: what sum of money has each?

$$A \text{ has } \frac{mnb-ma}{m-n} \text{ £ and } B \text{ has } \frac{nb-a}{m-n} \text{ £}.$$

15. Given the sum of two quantities $=a$, and the sum of $m$ times the former and $n$ times the latter $=b$, to find them.

$$\text{The former } = \frac{b-na}{m-n} \text{ and the latter } = \frac{ma-b}{m-n}.$$

16. Three magnitudes $A$, $B$, $C$ are such that the sum of $A$ and $B$ is $c$, that of $A$ and $C$ is $b$, and that of $B$ and $C$ is $a$: find them.

$$A = \frac{b+c-a}{2}, \quad B = \frac{a+c-b}{2} \text{ and } C = \frac{a+b-c}{2}.$$

17. Divide a given quantity $a$ into two parts so that the sum of their quotients by $m$ and $n$ respectively may $=b$.

$$\text{One part is } \frac{m(a-nb)}{m-n} \text{ and the other is } \frac{n(mb-a)}{m-n}.$$

18. Divide a given magnitude $a$ into three parts, so that the second may be $m$ times, and the third $n$ times, as great as the first.

$$\text{The parts are } \frac{a}{1+m+n}, \quad \frac{ma}{1+m+n} \text{ and } \frac{na}{1+m+n}.$$

19. Find two magnitudes whose difference is $a$ and the difference of whose squares is $b^2$.

The quantities are $\dfrac{b^2+a^2}{2a}$ and $\dfrac{b^2-a^2}{2a}$.

20. Divide the number $a$ into four parts, so that the first being increased by $b$, the second diminished by $b$, the third multiplied by $b$, and the fourth divided by $b$, the results may all be equal. The parts are

$$\frac{ab}{(1+b)^2}-b,\quad \frac{ab}{(1+b)^2}+b,\quad \frac{a}{(1+b)^2} \text{ and } \frac{ab^2}{(1+b)^2}.$$

21. Two pipes fill a cistern in $m$ and $n$ hours respectively: in what time will they fill it together?

The required time is $\dfrac{mn}{m+n}$ hours.

22. Find the time in which three persons can jointly perform a piece of work, when they can separately do it in $m$, $n$ and $p$ days. The required time is $\dfrac{mnp}{mn+mp+np}$ days.

23. Find two quantities whereof the former being increased by $a$ becomes $m$ times as great as the latter, and the latter being increased by $b$ becomes $n$ times as great as the former. The quantities are $\dfrac{a+mb}{mn-1}$ and $\dfrac{na+b}{mn-1}$.

24. Divide £$a$. among three persons, so that the first may have $m$ times as much as the second, and the third $n$ times as much as the first and second together.

The first has $\dfrac{ma}{(m+1)(n+1)}$ £, the second $\dfrac{a}{(m+1)(n+1)}$ £,

and the third $\dfrac{na}{n+1}$ £.

25. $A$ and $B$ can perform a piece of work in $m$ days, $A$ and $C$ in $n$ days, and $B$ and $C$ in $p$ days: in what times can they accomplish it individually and collectively?

$A$ can do it in $\dfrac{2mnp}{mp+np-mn}$ days, $B$ in $\dfrac{2mnp}{mn+np-mp}$ days, $C$ in $\dfrac{2mnp}{mn+mp-np}$ days; and $A$, $B$ and $C$ in $\dfrac{2mnp}{mn+mp+np}$ days.

## II. QUADRATIC EQUATIONS.

1. In $11x^2-44=5x^2+10$, $x=3$ and $-3$.

2. In $(x+2)^2=4x+5$, $x=1$ and $-1$.

3. In $\dfrac{x^2-12}{3}=\dfrac{x^2-4}{4}$, $x=6$ and $-6$.

4. In $x\sqrt{6+x^2}=1+x^2$, $x=\frac{1}{2}$ and $-\frac{1}{2}$.

5. In $\sqrt{x-a}=\sqrt{x+\sqrt{b^2+x^2}}$, $x=\pm\sqrt{a^2-b^2}$.

6. In $\dfrac{a}{x}+\dfrac{\sqrt{a^2-x^2}}{x}=\dfrac{x}{b}$, $x=\pm\sqrt{2ab-b^2}$.

7. In $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}=\dfrac{a}{x^2}$, $x=\pm\frac{1}{2}a\sqrt{3}$.

8. In $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x$, $x=\pm\sqrt{3}$.

9. In $\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=\dfrac{b}{c}$, $x=\pm\dfrac{a(b-c)}{2\sqrt{bc}}$.

10. In $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b$, $x=\pm\dfrac{a(\sqrt{b}\mp 1)^2}{2\sqrt{b}}$.

11. In $\dfrac{ax+1+\sqrt{a^2x^2-1}}{ax+1-\sqrt{a^2x^2-1}}=\dfrac{b^2x}{2}$, $x=\pm\dfrac{2}{b\sqrt{4a-b^2}}$.

12. In $\dfrac{\sqrt{a^2-x^2}-\sqrt{b^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\dfrac{c}{d}$,

$$x=\pm\sqrt{\frac{a^2(c-d)^2-b^2(c+d)^2}{2(c^2+d^2)}}.$$

13. In $x^2+20=12x$, $x=10$ and $2$.

14. In $x^2 - 14 = 13x$, $x = 14$ and $-1$.

15. In $x^2 + 2x - 35 = 0$, $x = 5$ and $-7$.

16. In $5x^2 - 24x = 5$, $x = 5$ and $-\frac{1}{5}$.

17. In $9x - 5x^2 = 2\frac{1}{4}$, $x = 1\frac{1}{2}$ and $\frac{3}{10}$.

18. In $\frac{1}{3}x^2 - \frac{1}{2}x = 9$, $x = 6$ and $-4\frac{1}{2}$.

19. In $\dfrac{4x}{5-x} - \dfrac{20-4x}{x} = 15$, $x = 4$ and $-1\frac{2}{3}$.

20. In $2x = 4 + \dfrac{6}{x}$, $x = 3$ and $-1$.

21. In $x - \dfrac{x^3-8}{x^2+5} = 2$, $x = 2$ and $\frac{1}{2}$.

22. In $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}$, $x = 2$ and $-3$.

23. In $\dfrac{x+4}{x-3} - \dfrac{2x-3}{x+4} = \dfrac{59}{8}$, $x = 4$ and $-2\frac{57}{67}$.

24. In $\dfrac{x+7}{x+11} - \dfrac{x+5}{x+12} = \dfrac{47}{306}$, $x = 6$ and $-9\frac{22}{47}$.

25. In $\dfrac{x-1}{x+1} + \dfrac{x+3}{x-3} = 2\dfrac{x+2}{x-2}$, $x = 5$ and $0$.

26. In $\dfrac{5x+36}{10x-81} + \dfrac{3}{25} = \dfrac{8x}{5x-8}$, $x = 14\frac{3}{5}$ and $\frac{24}{83}$.

27. In $\dfrac{3x}{x+2} - \dfrac{x-1}{6} = x - 9$, $x = 10$ and $-1\frac{4}{7}$.

28. In $x + \dfrac{24}{x-1} = 3x - 4$, $x = 5$ and $-2$.

29. In $\sqrt{2x+3} \times \sqrt{3x+7} = 12$, $x = 3$ and $-6\frac{5}{6}$.

30. In $2\sqrt{x} + \dfrac{2}{\sqrt{x}} = 5$, $x = 4$ and $\frac{1}{4}$.

31. In $\dfrac{x-4}{\sqrt{x}+2} = x - 8$, $x = 9$ and $4$.

32. In $3\sqrt{112-8x} = 19 + \sqrt{3x+7}$, $x = 6$ and $11\frac{523}{625}$.

33. In $\sqrt{x^2+\sqrt{x^4+22x^2+32x}}=x+2,\ x=\pm\sqrt{2}.$

34. In $(9+5\sqrt{3})x^2+(15+7\sqrt{3})x+6=0,$

$$x=2-\sqrt{3} \text{ and } 3-\sqrt{3}.$$

35. In $2x^2+\sqrt{x^2+9}=x^4-9,\ x=\pm\sqrt{\frac{3\pm\sqrt{41}}{2}}.$

36. In $\sqrt{4x^{\frac{2}{3}}+9}=4x^{\frac{1}{3}}-\sqrt{4x^{\frac{2}{3}}+5},\quad x=\pm 8.$

37. In $x^2-(a+b)x+ab=0,\ x=a$ and $b$.

38. In $ax^2-2a\sqrt{b}\,x=bx^2-ab,\ x=\frac{\sqrt{ab}}{\sqrt{a}\mp\sqrt{b}}.$

39. In $\sqrt{a+x}+\sqrt{x}=\frac{3a+4x}{\sqrt{a+x}},\ x=\frac{(-11\pm\sqrt{-7})a}{16}.$

40. In $a^3x^2+(1+b)a\sqrt{b}+a^2bx^2=\{a^3\sqrt{b}+(a+b)(1+b)\}x,$

$$x=\frac{a\sqrt{b}}{a+b} \text{ and } \frac{1+b}{a^2}.$$

41. In $x^2-2b\sqrt{a^2-ax+x^2}=ax-b^2,$

$$x=\frac{a}{2}\pm\sqrt{b^2\pm 2ab+\frac{a^2}{4}}.$$

42. In $2(x-a)\sqrt{a^2-x^2}+a^2+b^2=2ax,$

$$x=\frac{a\pm\sqrt{a^2-b^2}\pm\sqrt{b^2\mp 2a\sqrt{a^2-b^2}}}{2}.$$

43. In $\frac{\sqrt{a^2+ax+x^2}-\sqrt{a^2-ax+x^2}}{\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}}=\frac{a}{b},$

$$x=\frac{a^2+b^2}{4b}\pm\sqrt{\left(\frac{a^2+b^2}{4b}\right)^2-a^2}.$$

44. In $(a^{4m}+1)(x^{\frac{1}{2}}-1)^2=2(x+1),\ x=(a^{2m}\pm 1)^2.$

45. In $a^2b^2x^{\frac{1}{n}}-4a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{m+n}{2mn}}=(a-b)^2x^{\frac{1}{m}},$

$$x=\left(\frac{1}{\sqrt{a}}\pm\frac{1}{\sqrt{b}}\right)^{\frac{4mn}{m+n}}.$$

46. In $x^{\frac{p+q}{2pq}} - \frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2}\left(x^{\frac{1}{p}} + x^{\frac{1}{q}}\right) = 0,$

$$x = \left\{\frac{\sqrt{3a^2+b^2} - \sqrt{a^2+3b^2}}{\sqrt{3a^2+b^2} + \sqrt{a^2+3b^2}}\right\}^{\frac{2pq}{p+q}}.$$

47. In $\dfrac{\sqrt[m]{a+x}}{a} + \dfrac{\sqrt[m]{a+x}}{x} = \dfrac{\sqrt[m]{x}}{a}$, $x = \dfrac{a\,c^{\frac{m}{m-1}}}{a^{\frac{m}{m-1}} - c^{\frac{m}{m-1}}}$.

48. In $x^2 - 2x^{\frac{3}{2}} + 2x - \sqrt{x} = 6$, $x = 1, 4$ and $\frac{1}{2}(\pm\sqrt{-11} - 5)$.

49. In $x^2(x+4) + 2x(x+4) = 2 - (x+4)$, $x = -2 \pm \sqrt{3}$.

50. In $x^2 + \dfrac{36}{x^2} - 2x - \dfrac{12}{x} = 3$, $x = 2$, $3$ and $\dfrac{-3 \pm \sqrt{-15}}{2}$.

51. In $\sqrt{x} - \dfrac{8}{x} = \dfrac{7}{\sqrt{x} - 2}$, $x = 16$, $1$ and $\dfrac{1 \mp 3\sqrt{-7}}{2}$. Colenso p. xxxvi n.

52. In $\dfrac{\sqrt{1+x}}{1+\sqrt{1+x}} = \dfrac{\sqrt{1-x}}{1-\sqrt{1-x}}$, $x = \pm\dfrac{\sqrt{3}}{2}$ and $0$.

53. In $\sqrt{a+x} + \sqrt{a-x} = \dfrac{x}{\sqrt{6}}$, $x = \pm 2\sqrt{6a - 36}$.

54. In $x^{4m} - 2x^{3m} + x^m = a$, $x = \sqrt[m]{\dfrac{1 \pm \sqrt{3 \pm 2\sqrt{4a+1}}}{2}}$.

55. In $x^2(x-2)^2 + 5x^2(x-2) = 2(x-1)$, $x = \pm\sqrt{3} - 1$.

56. In $(x-4)^2 + 2(x-4) = \dfrac{2}{x} - 1$, $x = 2$ and $2 \pm \sqrt{3}$.

57. In $\dfrac{x^2}{4} = \dfrac{x-12}{x^2-18}$, $x = \pm 4$ and $\pm 2$.

58. In $2x^3 - x^2 = 1$, $x = 1$ and $\dfrac{-1 \pm \sqrt{-7}}{4}$.

59. In $x + 1 = \dfrac{2}{\sqrt{x}}$, $x = 1$ and $\dfrac{-3 \pm \sqrt{-7}}{2}$.

60. In $x - 3 = \dfrac{3 + 4\sqrt{x}}{x}$, $x = \dfrac{7 \pm \sqrt{13}}{2}$ and $\dfrac{-1 \mp \sqrt{-3}}{2}$.

61. In $x^2 - \frac{2}{3x} = 1\frac{4}{9}$, $x = -\frac{2}{3}$ and $\frac{1 \pm \sqrt{10}}{3}$.

62. In $\frac{x}{2} = \frac{24 + 7\sqrt{x}}{x+1}$, $x = 9$, $4$ and $\frac{-15 \pm \sqrt{-31}}{2}$.

## Problems.

1. Find two numbers, one of which is $\frac{3}{5}$ths of the other, so that the difference of their squares may be the square of 16.

The numbers are $\pm 20$ and $\pm 12$.

2. What two magnitudes are those whose product is $a$ and quotient $b$?

The magnitudes are $\pm \sqrt{ab}$ and $\pm \sqrt{\frac{a}{b}}$.

3. To find two numbers whose difference is 8 and product 128.

The numbers are $\pm 8$ and $\pm 16$.

4. Determine two magnitudes whose difference is $\frac{1}{6}$, and the sum of whose squares is $(\frac{5}{6})^2$.

The magnitudes are $\pm \frac{1}{2}$ and $\pm \frac{2}{3}$.

5. It is required to find two magnitudes whose difference is $b$, such that if $a^2$ be divided by each of them, the difference of the quotients shall be $c$.

The magnitudes are

$$\frac{-bc \pm \sqrt{4a^2bc + b^2c^2}}{2c} \text{ and } \frac{bc \pm \sqrt{4a^2bc + b^2c^2}}{2c}.$$

6. Find three magnitudes, the products of each two of which are $p$, $q$ and $r$ respectively.

The magnitudes are $\pm \sqrt{\frac{pq}{r}}$, $\pm \sqrt{\frac{pr}{q}}$ and $\pm \sqrt{\frac{qr}{p}}$.

7. Find two numbers each of which together with six times its reciprocal shall be equal to 5.

The numbers required are 2 and 3.

8. The reckoning of a party at a tavern was £3. 12s.; but in consequence of two of them having no money, each of the rest paid 6*d.* more than he otherwise should have done: required their number. The number is 18.

9. Divide 120 into two parts so that the sum of the quotients arising from dividing each by the other may be $2\frac{4}{35}$.

The parts required are 50 and 70.

10. Divide the given magnitude $a$ into two parts so that the product of the whole and one of the parts shall be equal to the square of the other part.

The parts are $\frac{1}{2}a\left\{\sqrt{5}-1\right\}$ and $\frac{1}{2}a\left\{3-\sqrt{5}\right\}$.

11. To find two magnitudes whose product is $a^2$ and the sum of whose squares is $b^2$.

The magnitudes are

$$\pm\sqrt{\frac{b^2\pm\sqrt{b^4-4a^4}}{2}} \text{ and } \pm\sqrt{\frac{b^2\mp\sqrt{b^4-4a^4}}{2}}.$$

12. Find two magnitudes such that the first together with twice the second may be 23, and the sum of their squares 130.

The quantities are 9 and 7, or $\frac{1}{5}$ and $\frac{57}{5}$.

13. Given the sum and difference of the squares of two magnitudes equal to $a^2$ and $b^2$ respectively, to find them.

The magnitudes are $\pm\sqrt{\dfrac{a^2+b^2}{2}}$ and $\pm\sqrt{\dfrac{a^2-b^2}{2}}$.

14. Divide the given quantity $a$ into two parts that the sum of their square roots may be $b$.

The parts are $\dfrac{a\pm\sqrt{2ab-b^2}}{2}$ and $\dfrac{a\mp\sqrt{2ab-b^2}}{2}$.

15. To find two magnitudes whose sum is $a$ and the sum of whose cubes is $b^3$.

The magnitudes are $\dfrac{a}{2}\pm\sqrt{\dfrac{b^3}{3a}-\dfrac{a^2}{12}}$ and $\dfrac{a}{2}\mp\sqrt{\dfrac{b^3}{3a}-\dfrac{a^2}{12}}$.

16. Given $a$ the product of two magnitudes, and $b$ the difference of the products arising from multiplying them by $c$ and $d$ respectively: to find them.

The magnitudes are $\dfrac{\pm\sqrt{b^2+4acd}+b}{2c}$ and $\dfrac{\pm\sqrt{b^2+4acd}-b}{2d}$.

17. Given the sum of two quantities $=a$ and the sum of their fourth powers $=b^4$; to find them.

The quantities are

$$\frac{a\pm\sqrt{-3a^2+\sqrt{8(a^4+b^4)}}}{2} \text{ and } \frac{a\mp\sqrt{-3a^2+\sqrt{8(a^4+b^4)}}}{2}.$$

18. Find three magnitudes, when the product of the first and second is $a$, the product of the first and third is $b$, and the sum of the squares of the second and third is $c$.

The magnitudes are

$$\pm\sqrt{\frac{a^2+b^2}{c}},\ \pm a\sqrt{\frac{c}{a^2+b^2}} \text{ and } \pm b\sqrt{\frac{c}{a^2+b^2}}.$$

19. It is required to divide each of the numbers 11 and 17 into two parts, so that the product of the first parts of each may be 45, and of the second 48.

The parts of 11 are 5 and 6, and those of 17 are 9 and 8.

20. Divide each of the numbers 21 and 30 into two parts, so that the first part of 21 may be three times as great as the first part of 30, and that the sum of the squares of the remaining parts may be 585.

The parts of 21 are 18 and 3, and those of 30 are 6 and 24.

21. To divide each of the numbers 19 and 29 into two parts, so that the difference of the squares of the first parts of each may be 72, and the difference of the squares of the remaining parts 180.

The parts of 19 are 7 and 12, and those of 29 are 11 and 18.

22. To divide each of two magnitudes $a$ and $b$ into two parts, that $m$ times one part of $a$ may be equal to $n$ times one part of $b$, and that the product of the remaining parts may be $c^2$.

The parts of $a$ are $\dfrac{ma+nb\pm\sqrt{(ma-nb)^2+4mnc^2}}{2m}$

and $\dfrac{ma-nb\mp\sqrt{(ma-nb)^2+4mnc^2}}{2m}$;

and those of $b$ are $\dfrac{ma+nb\pm\sqrt{(ma-nb)^2+4mnc^2}}{2n}$

and $\dfrac{-ma+nb\mp\sqrt{(ma-nb)^2+4mnc^2}}{2n}$.

**23.** *A* and *B* start from two places *C* and *D* at the same time, *A* from *C* intending to pass through *D*, and *B* from *D* travelling the same way: when *A* overtakes *B* it is found that they had together travelled 30 miles; that *A* had passed through *D* four hours before, and that *B* was nine hours journey from *C*: find the distance between *C* and *D*, and the rates of travelling of *A* and *B*.

The distance between *C* and *D* is 6 miles, and the rates of travelling of *A* and *B* are 3 and 2 miles per hour.

**24.** It is required to divide each of the three numbers 17, 23 and 38 into two parts, so that the product of one part of 17 and one part of 23 may be 63; the product of the other part of 17 and one part of 38 may be 180, and the product of the remaining parts of 23 and 38 may be 280.

The parts of 17 are 7 and 10, those of 23 are 9 and 14, and those of 38 are 18 and 20.

## III. SIMULTANEOUS EQUATIONS.

1. In $x+y=9$ and $3x+5y=35$, $x=5$, $y=4$.

2. In $2x+3y=18$ and $3x-2y=1$, $x=3$, $y=4$.

3. In $2x-9y=11$ and $3x-12y=15$, $x=1$, $y=-1$.

4. In $3x-7y=7$ and $11x+5y=87$, $x=7$, $y=2$.

5. In $9x-4y=8$ and $13x+7y=101$, $x=4$, $y=7$.

6. In $x-\dfrac{y-2}{7}=5$ and $4y-\dfrac{x+10}{3}=3$, $x=5$, $y=2$.

7. In $\dfrac{4x+3y}{6}=8$ and $\dfrac{7y-3x}{2}-y=11$, $x=6$, $y=8$.

8. In $2x-\dfrac{y-3}{5}=4$ and $3y+\dfrac{x-2}{3}=9$, $x=2$, $y=3$.

9. In $2x+.4y=1.2$ and $3.4x-.02y=.01$, $x=.02$, $y=2.9$.

10. In $\dfrac{2x-y}{7}+3x=2y-6$ and $\dfrac{y+3}{5}+\dfrac{y-x}{6}=2x-8$, $x=6$, $y=12$.

11. In $\frac{1}{40}(4x+5y)=x-y$ and $\frac{1}{3}(2x-y)+29=\frac{1}{2}$, $x=\frac{1}{4}$, $y=\frac{}{5}$.

12. In $2y-\dfrac{x-6y+1}{7}=\dfrac{x-3}{2}$ and $\dfrac{x-5y+8}{9}=\dfrac{3x-13y}{7}$, $x=11$, $y=2$.

13. In $\dfrac{x-y}{5}=2$ and $\dfrac{3x-7y}{3}=\dfrac{2x+y+1}{5}$, $x=13$, $y=3$.

14. In $\dfrac{4x+2y}{11}=6-\dfrac{5y-3x}{4}$ and $\dfrac{8y-10}{3}=\dfrac{5x+3y}{6}+5$, $x=3$, $y=5$.

15. In $\dfrac{5x-6y}{13}+3x=4y-2$ and $\dfrac{5x+6y}{6}-\dfrac{3x-2y}{4}=2y-2$, $x=6$, $y=5$.

16. In $\dfrac{x-2}{5}-\dfrac{10-x}{3}=\dfrac{y-10}{4}$ and $\dfrac{2y+4}{3}-\dfrac{2x+y}{8}=\dfrac{x+13}{4}$, $x=7$, $y=10$.

17. In $(x+5)(y+7)=(x+1)(y-9)+112$ and $2x+10=3y+1$, $x=3$, $y=5$.

18. In $3x-\dfrac{151-16x}{4y-1}=\dfrac{9xy-110}{3y-4}$ and $3x+6y+1=\dfrac{6x^2+130-24y^2}{2x-4y+3}$, $x=9$, $y=2$.

19. In $\dfrac{4x-2y+3}{3}-\dfrac{18-x+5y}{7}=\dfrac{x}{4}-\dfrac{y}{5}-\dfrac{1}{7}-7\tfrac{7}{10}$ and $(2x-y+15)\left(\dfrac{y}{4}-\dfrac{x}{3}+\dfrac{1}{12}\right)=(y-2x+15)\left(\dfrac{x}{3}-\dfrac{y}{4}+\dfrac{3}{4}\right)$, $x=18,\ y=24$.

20. In $\dfrac{3x-2y}{3}+1+\dfrac{11y-10}{8}=\dfrac{4x-3y+5}{7}+\dfrac{45-x}{5}$ and $45-\dfrac{4x-2}{3}=\dfrac{55x+71y+1}{18}$, $x=5,\ y=6$.

21. In $\dfrac{1+x}{1-y}+\dfrac{1+y}{1-x}=\dfrac{9}{13}$ and $\dfrac{1+x}{1+y}+\dfrac{1-y}{1-x}=\dfrac{4}{13}$, $x=1$ or $1\tfrac{72}{169}$, $y=\tfrac{5}{13}$ or $1$.

22. In $\sqrt{y}-\sqrt{a-x}=\sqrt{y-x}$ and $2\sqrt{y-x}=3\sqrt{a-x}$, $x=\tfrac{4}{5}a,\ y=\tfrac{5}{4}a$.

23. In $\dfrac{4x-8y+5}{2}=\dfrac{10x^2-12y^2-14xy+2x}{5x+3y+3}+2$ and $2\sqrt{6+x}=3\sqrt{6-y}$, $x=3,\ y=2$.

24. In $3x+\tfrac{2}{3}\sqrt{xy^2+9x^2y}=(x-\tfrac{1}{3})y$ and $18x-2y=xy$, $x=4,\ y=12$.

25. In $x^2+xy+y^2=52$ and $xy-x^2=8$, $x=\pm 2,\ y=\pm 6$.

26. In $x^2-2xy-y^2=31$ and $\tfrac{1}{2}x^2+2xy-y^2=101$, $x=\pm 10$, $y=\pm 3$.

27. In $y^2+y+17x=54$ and $\sqrt{x^2+2y^2}+x=8$, $x=2$ or $2\tfrac{64}{81}$, $y=\pm 4$ or $\pm\tfrac{28}{81}$.

28. In $2(x-y)=11$ and $xy=20$, $x=8,\ y=2\tfrac{1}{2}$.

29. In $x^4+y^4=97$ and $x+y=5$, $x=2,\ y=3$.

30. In $x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x$ and $x^{\frac{1}{2}}+y^{\frac{1}{3}}=x$, $x=4$ or $1$, $y=8$.

31. In $\sqrt{5\sqrt{x}+5\sqrt{y}}+\sqrt{y}=10-\sqrt{x}$ and $x^{\frac{5}{2}}+y^{\frac{5}{2}}=275$, $x=9$ or $4$, $y=4$ or $9$.

32. In $x^2 + 10x + y = 119 - 2(x+5)\sqrt{y}$ and $x + 2y = 13$, $x = 5$ or $8\frac{1}{2}$, $y = 4$ or $2\frac{1}{4}$.

33. In $y - y^{\frac{1}{2}} = 16 - x$ and $28 - y = x + 4x^{\frac{1}{2}}$, $x = 4$, $y = 16$.

34. In $x - y = 2$ and $\frac{x}{y} - \frac{y}{x} = 1\frac{1}{15}$, $x = 5$ or $\frac{3}{4}$, $y = 3$ or $-1\frac{1}{4}$.

35. In $2x + y = 26 - 7\sqrt{2x + y + 4}$ and $\frac{2x + \sqrt{y}}{2x - \sqrt{y}} = \frac{16}{15} + \frac{2x - \sqrt{y}}{2x + \sqrt{y}}$, $x = 2$ or $-10$, $y = 1$ or $25$.

36. In $\sqrt{x} + \sqrt{y} = 6$ and $x + y = 20$, $x = 16$ or $4$, $y = 4$ or $16$.

37. In $x + y + \sqrt{xy} = 28$ and $x^2 + y^2 + xy = 336$, $x = \pm 16$, $y = \pm 4$.

38. In $\frac{1}{10}(x^2 + y^2) = \frac{1}{3}(x + y)$ and $xy = 8$, $x = 4$ or $2$ or $\frac{1}{3}(-4 \pm 2\sqrt{-14})$, $y = 2$ or $4$ or $\frac{1}{3}(-4 \mp 2\sqrt{-14})$.

39. In $\frac{x}{x+y} - \frac{y}{x} = \frac{x^2 - y^2}{18}$ and $\frac{x}{y} - \frac{x+y}{x} = \frac{y}{x}$, $x = \pm 2$, $y = \pm 1$.

40. In $x^5 - y^5 = 19$ and $x - y = 1$, $x = 3$ or $-2$, $y = 2$ or $-3$.

41. In $x^3 + y^3 = 189$ and $x^2y + xy^2 = 180$, $x = 4$ or $5$, $y = 5$ or $4$.

42. In $x^4 - x^2 + y^4 - y^2 = 84$ and $x^2 + x^2y^2 + y^2 = 49$,

$$x = \pm 3 \text{ or } \pm\tfrac{1}{2}\{\sqrt{-14 \pm 6\sqrt{7}} + \sqrt{-14 \mp 6\sqrt{7}}\},$$
$$y = \pm 2 \text{ or } \pm\tfrac{1}{2}\{\sqrt{-14 \pm 6\sqrt{7}} - \sqrt{-14 \mp 6\sqrt{7}}\}.$$

43. In $x + y = 6$ and $x^5 + y^5 = 1056$, $x = 4$ or $2$ or $3 \pm \sqrt{-19}$, $y = 2$ or $4$ or $3 \mp \sqrt{-19}$.

44. In $x^{\frac{4}{3}} + y^{\frac{2}{5}} = 85$ and $x^{\frac{2}{3}} + y^{\frac{1}{5}} = 11$, $x = \pm 27$ or $\pm 2\sqrt{2}$, $y = 32$ or $59049$.

45. In $\sqrt{y} - \sqrt{y - x} = \sqrt{20 - x}$ and $2\sqrt{\frac{y - x}{20 - x}} = 3$, $x = 16$ or $40$, $y = 25$ or $-5$.

46. In $\frac{y}{x}-\frac{81}{xy}=(2y+18)\frac{\sqrt{x}}{y}$ and $y+3x^{\frac{3}{2}}=9+x^{\frac{3}{2}}\sqrt{y}$, $x=4$, $y=25$.

47. In $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{61}{\sqrt{xy}}+1$ and $\sqrt[4]{x^3y}+\sqrt[4]{xy^3}=78$, $x=81$ or $16$, $y=16$ or $81$.

48. In $\sqrt{\frac{x+y^2}{4x}}+\frac{y}{\sqrt{y^2+x}}=\frac{y^2}{4}\sqrt{\frac{4x}{y^2+x}}$ and $\frac{\sqrt{x}+\sqrt{x-y-1}}{\sqrt{x}-\sqrt{x-y-1}}=y+1$, $x=4$ or $\frac{1}{4}$, $y=2$ or $-1$.

49. In $\frac{y}{2x}+\frac{2}{3}\frac{y-\sqrt{x-1}}{y^2-2\sqrt{x^2-1}}=\frac{\sqrt{x+1}}{x}$ and $\frac{1}{4}y^4=y^2x-1$, $x=1\frac{1}{4}$, $y=\pm 2$.

50. In $\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}=4\frac{1}{4}-\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$ and $x(x+y)=52-\sqrt{x^2+xy+4}$, $x=\pm 5$, $y=\pm 4$.

51. In $\frac{x+y+\sqrt{x^2-y^2}}{x+y-\sqrt{x^2-y^2}}=\frac{9}{8y}(x+y)$ and $(x^2+y)^2+x-y=2x(x^2+y)+506$, $x=5$ or $-4\frac{3}{5}$, $y=3$ or $-2\frac{19}{25}$.

52. In $\frac{9}{8}\frac{\sqrt[3]{x+y}}{y}+\frac{9}{8}\frac{\sqrt[3]{x+y}}{x}=\frac{8}{7}$ and $\frac{7}{4}\frac{\sqrt[3]{x-y}}{y}-\frac{7}{4}\frac{\sqrt[3]{x-y}}{x}=\frac{1}{9}$, $x=\pm 4\frac{1}{2}$ or $\pm 3\frac{1}{2}$, $y=\pm 3\frac{1}{2}$ or $\pm 4\frac{1}{2}$.

53. In $\frac{y}{x}\sqrt{\frac{x}{y}}+\sqrt{\frac{x}{2y}}\sqrt[4]{\frac{y^3}{x^3}}=3$ and $\frac{3x^2}{y}-\frac{x}{4\sqrt{y}}=11\frac{1}{2}$, $x=16$, $y=64$.

54. In $5y+\frac{1}{5}\sqrt{x^2-15y-14}=\frac{1}{3}x^2-36$ and $\frac{x^2}{8y}+\frac{2x}{3}=\sqrt{\frac{x^3}{3y}+\frac{x^2}{4}}-\frac{y}{2}$, $x=12$ or $-9\frac{1}{2}$, $y=2$ or $-1\frac{7}{12}$.

55. In $2x+\sqrt{x^2-y^2}=\frac{14}{y}\left\{\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}\right\}$ and $\left(\frac{x+y}{2}\right)^{\frac{3}{2}}+\left(\frac{x-y}{2}\right)^{\frac{3}{2}}=9$, $x=5$, $y=3$.

56. In $3x-x\sqrt{\frac{5}{4}x^2-2y+8}=2-y$ and $\frac{\sqrt{x+y}}{2x}-\frac{3x}{4}=\frac{2x-3}{\sqrt{x+y}}-\frac{3y}{2x}$, $x=2$ or $-\frac{34}{31}$, $y=2$ or $\frac{3554}{961}$.

57. In $x^2y-4=4x^{\frac{1}{2}}y-\frac{1}{4}y^3$ and $x^{\frac{3}{2}}-3=x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}})$, $x=1$, $y=4$.

58. In $2a(y-3b)=b(2a-x)$ and $ay=bx$, $x=\frac{8}{3}a$, $y=\frac{8}{3}b$.

59. In $x+ay=b$ and $ax-by=c$, $x=\frac{ac+b^2}{a^2+b}$, $y=\frac{ab-c}{a^2+b}$.

60. In $\frac{1}{x}+\frac{1}{y}=m$ and $\frac{1}{x^2}+\frac{1}{y^2}=n^2$, $x=\frac{2}{m+\sqrt{2n^2-m^2}}$, $y=\frac{2}{m-\sqrt{2n^2-m^2}}$.

61. In $a(3a-x)=b(b+y)$ and $ax+2by=c$, $x=\frac{2b^2-6a^2+c}{3a}$, $y=\frac{3a^2-b^2+c}{3b}$.

62. In $(a^2-b^2)(3x+5y)=(4a-b)2ab$ and $a^2x-\frac{ab^2c}{a+b}+(a+b+c)by=b^2x+(a+2b)ab$, $x=\frac{ab}{a-b}$, $y=\frac{ab}{a+b}$.

63. In $bx=cy$ and $x^3-y^3=d$, $x=c\sqrt[3]{\frac{d}{c^3-b^3}}$, $y=b\sqrt[3]{\frac{d}{c^3-b^3}}$.

64. In $x(bc-xy)=y(xy-ac)$ and $xy(ay+bx-xy)=abc(x+y-c)$, $x=\pm\sqrt{ac}$, $y=\pm\sqrt{bc}$.

65. In $x+y=a(x-y)$ and $x^2+y^2=b^2$,

$$x=\pm\frac{(a+1)b}{\sqrt{2(a^2+1)}},\quad y=\pm\frac{(a-1)b}{\sqrt{2(a^2+1)}}.$$

66. In $(x-y)(x^2-y^2)=a^3$ and $(x+y)(x^2+y^2)=b^3$,

$$x=\frac{\sqrt{2b^3-a^3}\pm\sqrt{a^3}}{2\sqrt[6]{2b^3-a^3}},\quad y=\frac{\sqrt{2b^3-a^3}\mp\sqrt{a^3}}{2\sqrt[6]{2b^3-a^3}}.$$

67. In $x^2-y^2=a^2$ and $(x+y+b)^2+(x-y+b)^2=c^2$,

$$x=\tfrac{1}{2}(-b\pm\sqrt{2a^2-b^2+c^2}),\quad y=\pm\tfrac{1}{2}\sqrt{c^2-2a^2\mp 2b\sqrt{2a^2-b^2+c^2}}.$$

68. In $x+y=xy$ and $x+y+x^2+y^2=a$,

$$x=\tfrac{1}{4}(1\pm\sqrt{4a+1}+\sqrt{4a-6\mp 6\sqrt{4a+1}}),$$
$$y=\tfrac{1}{4}(1\pm\sqrt{4a+1}-\sqrt{4a-6\mp 6\sqrt{4a+1}}).$$

69. In $\frac{x}{a}+\frac{y}{b}=1-\frac{x}{c}$ and $\frac{y}{a}+\frac{x}{b}=1+\frac{y}{c}$,

$$x=\frac{abc(ac+ab-bc)}{a^2b^2+a^2c^2-b^2c^2},\quad y=\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}.$$

70. In $x-ay+a^2z-a^3=0$, $x-by+b^2z-b^3=0$ and $x-cy+c^2z-c^3=0$, $x=abc$, $y=ab+ac+bc$, $z=a+b+c$.

71. In $x+y+z=14$, $x^2+y^2+z^2=84$ and $xz=y^2$, $x=2$, $y=4$, $z=8$.

72. In $xy=a(x+y)$, $xz=b(x+z)$ and $yz=c(y+z)$,

$$x=\frac{2abc}{ac+bc-ab},\quad y=\frac{2abc}{ab+bc-ac},\quad z=\frac{2abc}{ab+ac-bc}.$$

73. In $x(y+z)=a^2$, $y(x+z)=b^2$ and $z(x+y)=c^2$,

$$x=\pm\sqrt{\frac{(a^2-c^2+b^2)(a^2-b^2+c^2)}{2(c^2-a^2+b^2)}},$$

$$y=\pm\sqrt{\frac{(b^2-c^2+a^2)(b^2-a^2+c^2)}{2(a^2-b^2+c^2)}},$$

$$z=\pm\sqrt{\frac{(c^2-a^2+b^2)(c^2-b^2+a^2)}{2(b^2-c^2+a^2)}}.$$

74. In $xyz=a^2(x+y)$, $xyz=b^2(y+z)$ and $xyz=c^2(x+z)$,

$$x=\pm abc\sqrt{\frac{2(a^2b^2+b^2c^2-a^2c^2)}{(a^2b^2+a^2c^2-b^2c^2)(b^2c^2+a^2c^2-a^2b^2)}},$$

$$y=\pm abc\sqrt{\frac{2(b^2c^2+a^2c^2-a^2b^2)}{(a^2b^2+a^2c^2-b^2c^2)(a^2b^2+b^2c^2-a^2c^2)}},$$

$$z=\pm abc\sqrt{\frac{2(a^2b^2+a^2c^2-b^2c^2)}{(a^2b^2+b^2c^2-a^2c^2)(b^2c^2+a^2c^2-a^2b^2)}}.$$

75. In $x(x+y+z)=a^2$, $y(x+y+z)=b^2$, and $z(x+y+z)$ $=c^2$, $x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$, $y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$,

$$z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$$

76. In $xy=a$, $xz=b$, $xu=c$ and $xyzu=d$,

$$x=\pm\sqrt{\frac{abc}{d}},\ y=\pm\sqrt{\frac{ad}{bc}},\ z=\pm\sqrt{\frac{bd}{ac}},\ u=\pm\sqrt{\frac{cd}{ab}}.$$

77. In
$$\begin{aligned}
&u+ax+a^2y+a^3z+a^4=0,\\
&u+bx+b^2y+b^3z+b^4=0,\\
&u+cx+c^2y+c^3z+c^4=0,\\
&u+dx+d^2y+d^3z+d^4=0,
\end{aligned}$$
$u=abcd$, $x=-abc-abd-acd-bcd$, $y=ab+ac+ad$ $+bc+bd+cd$, $z=-a-b-c-d$.

78. Given $u+x+y+z=a$, $ux+uy+uz+xy+xz$ $+yz=b^2$, $uxy+uxz+uyz+xyz=c^3$ and $uxyz=d^4$, to eliminate $x$, $y$ and $z$.

The final equation is $u^4-au^3+b^2u^2-c^3u+d^4=0$.

79. In the $m$ equations,
$$\begin{aligned}
&x_2+x_3+x_4+\&c.+x_m=a_1,\\
&x_1+x_3+x_4+\&c.+x_m=a_2,\\
&x_1+x_2+x_4+\&c.+x_m=a_3,\\
&\&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&x_1+x_2+x_3+\&c.+x_{m-1}=a_m;
\end{aligned}$$

$$x_1 = \frac{1}{m-1}\{a_2 + a_3 + a_4 + \&c. + a_m - (m-2)\,a_1\},$$

$$x_2 = \frac{1}{m-1}\{a_1 + a_3 + a_4 + \&c. + a_m - (m-2)\,a_2\},$$

$$x_3 = \frac{1}{m-1}\{a_1 + a_2 + a_4 + \&c. + a_m - (m-2)\,a_3\},$$

&c.................................................

$$x_m = \frac{1}{m-1}\{a_1 + a_2 + a_3 + \&c. + a_{m-1} - (m-2)\,a_m\}.$$

80. If $X = ax + a_1x_1 + a_2$, $Y = bx + b_1x_1 + b_2$ and $Z = cx + c_1x_1 + c_2$, and also $aX + bY + cZ = 0$ and $a_1X + b_1Y + c_1Z = 0$; then will $X^2 + Y^2 + Z^2$

$$= \frac{\{a_2(bc_1 - b_1c) + b_2(a_1c - ac_1) + c_2(ab_1 - a_1b)\}^2}{(bc_1 - b_1c)^2 + (a_1c - ac_1)^2 + (ab_1 - a_1b)^2}.$$

## Problems.

1. The sum of two numbers is 17, and the product of one of them and its excess above the other is 55: find them.

The numbers are 11 and 6.

2. Divide the given magnitude $a$ into two parts, so that the sum of their squares may be equal to $m$ times their product.

The parts are

$$\frac{a}{2}\left\{\frac{\sqrt{m+2} \pm \sqrt{m-2}}{\sqrt{m+2}}\right\} \text{ and } \frac{a}{2}\left\{\frac{\sqrt{m+2} \mp \sqrt{m-2}}{\sqrt{m+2}}\right\}.$$

3. Find a fraction such that if its numerator be increased and denominator be diminished by 1, the result is $\frac{4}{5}$; but if its numerator be diminished and denominator be increased by 1, the result is $\frac{1}{2}$. The fraction is $\frac{7}{11}$.

4. Find two numbers whose product is 48, and the difference of whose squares is 28. The numbers are 6 and 8.

5. Given the difference of two quantities $= a$, and the difference between the sum of their squares and their product $= b^2$, to find them.

The quantities are $\frac{1}{2}(\sqrt{4b^2-3a^2}+a)$ and $\frac{1}{2}(\sqrt{4b^2-3a^2}-a)$.

6. What two magnitudes are those whose sum, quotient and difference of their squares, are equal to each other?

The magnitudes are $\frac{1}{2}(2+\sqrt{2})$ and $\frac{1}{2}\sqrt{2}$.

✓ 7. Find two magnitudes so that their product, the difference of their squares and the quotient of their cubes, may all be equal to one another.

The magnitudes are $\frac{1}{2}(3+\sqrt{5})$ and $\frac{1}{2}(1+\sqrt{5})$.

8. There are two magnitudes whose product is equal to the difference of their squares, and the sum of their squares is also equal to the difference of their cubes: find them.

The magnitudes are $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$.

9. $A$ and $B$ possess together $a$£, and it is found that after having spent an $m^{\text{th}}$ and $n^{\text{th}}$ part respectively, they have equal sums remaining: required the property of each.

$$A \text{ has } \frac{m(n-1)a}{2mn-m-n}\text{£} \quad \text{and } B \text{ has } \frac{n(m-1)a}{2mn-m-n}\text{£}.$$

10. The product of two magnitudes together with four times their sum is $51\frac{2}{3}$, and the sum of their squares diminished by four times their sum is $4\frac{97}{144}$: it is required to find them.

The magnitudes are $3\frac{1}{4}$ and $5\frac{1}{3}$.

11. Find two numbers whose sum added to the sum of their squares is equal to 62, and whose difference subtracted from the difference of their squares gives 40.

The numbers are 7 and 2.

12. Required two numbers whose sum multiplied by the sum of their squares is 272, and whose difference multiplied by the difference of their squares is 32. The numbers are 5 and 3.

13. Find two quantities, such that if one of them be increased by $a$ and $b$ and the other by $c$ and $d$, the products of the corresponding sums shall exceed the product of the quantities themselves by $e^2$ and $f^2$.

The quantities are

$$\frac{af^2 - be^2 + ab(c-d)}{ad-bc} \text{ and } \frac{de^2 - cf^2 - cd(a-b)}{ad-bc}.$$

14. Divide each of the numbers 21 and 35 into two parts, so that the difference of the squares of the first parts may be 57 and that of the second parts 407. They are 8 and 13, and 11 and 24.

15. Find two numbers such that the sum of their squares multiplied by the less and divided by the greater is $83\frac{1}{5}$, and the difference of their squares multiplied by the greater and divided by the less is 1920. The numbers are 20 and 4.

16. Find two quantities when the difference of their cube roots is $a$, and the cube root of their difference is $b$.

The magnitudes are

$$\left\{\sqrt{\frac{4b^3 - a^3}{12a}} + \frac{a}{2}\right\}^3 \text{ and } \left\{\sqrt{\frac{4b^3 - a^3}{12a}} - \frac{a}{2}\right\}^3.$$

17. Find two numbers whose sum is 5, such that the product of the sums of their squares and cubes may be 455.

The numbers are 3 and 2.

18. Find three magnitudes, the sums of each two of which are $a$, $b$ and $c$.

The magnitudes are $\frac{1}{2}(a+b-c)$, $\frac{1}{2}(a+c-b)$ and $\frac{1}{2}(b+c-a)$.

19. Find three magnitudes whose products taken two and two together are $a^2$, $b^2$ and $c^2$.

The magnitudes are $\pm\dfrac{ab}{c}$, $\pm\dfrac{ac}{b}$ and $\pm\dfrac{bc}{a}$.

20. $A$'s money together with twice that of $B$ and $C$ amounts to 1050£; $B$'s together with thrice that of $A$ and $C$ to 1400£, and $C$'s together with four times that of $A$ and $B$ to 1650£: required the money of each.

$A$ has 150£, $B$ has 200£ and $C$ has 250£.

21. Required four magnitudes whose products taken three and three together are $a^3$, $b^3$, $c^3$ and $d^3$.

The magnitudes are $\frac{abc}{d^2}$, $\frac{abd}{c^2}$, $\frac{acd}{b^2}$ and $\frac{bcd}{a^2}$.

22. There are four numbers such that if each be multiplied by their sum, the products are 252, 504, 396 and 144: it is required to find them. The numbers are 7, 14, 11 and 4.

23. Find four quantities such that the first with half of the rest, the second with a third of the rest, the third with a fourth of the rest, and the fourth with a fifth of the rest, may each be equal to $a$. The quantities are $\frac{1}{37}a$, $\frac{19}{37}a$, $\frac{25}{37}a$ and $\frac{28}{37}a$.

24. The sum of four numbers is 52: the sum of the products of the first and second and third and fourth is 360: of the first and third and second and fourth 315: and of the first and fourth and second and third 280: it is required to find them.

The numbers are 21, 14, 11 and 6.

25. Of $m$ quantities the continued products of every $m-1$ being $a$, $b$, $c$, $d$, &c. it is required to find them.

The quantities are

$$\frac{(abcd\ \&c.)^{\frac{1}{m-1}}}{a},\ \frac{(abcd\ \&c.)^{\frac{1}{m-1}}}{b},\ \frac{(abcd\ \&c.)^{\frac{1}{m-1}}}{c},\ \&c.$$

26. Given the sum of two numbers $=s$ and their product $=p$, find the sums of their squares, cubes, &c.

The sum of their squares $= s^2 - 2p$.

The sum of their cubes $= s^3 - 3ps$.

The sum of their fourth powers $= s^4 - 4ps^2 + 2p^2$.

The sum of their fifth powers $= s^5 - 5ps^3 + 5p^2 s$.

## CHAP. VII.

### MISCELLANEOUS EXAMPLES.

1. Prove, by the method of indeterminate coefficients, that

$$\frac{1+x}{1-x-x^2} = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + 13x^5 + \&c.$$

2. $$\frac{1+x}{(1-x)^2} = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + 11x^5 + \&c.$$

3. $\dfrac{1+x}{(1-x)^3} = 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + 5^2 x^4 + 6^2 x^5 + \text{\&c.}$

4. $\dfrac{x^m - a^m}{x - a} = x^{m-1} + a x^{m-2} + a^2 x^{m-3} + \text{\&c.} + a^{m-1}.$

5. $\sqrt{1+x+x^2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{3}{16}x^3 - \text{\&c.}$

6. $\sqrt{\dfrac{a+x}{a-x}} = \left(1+\dfrac{x}{a}\right) + \dfrac{x^2}{2a^2}\left(1+\dfrac{x}{a}\right)$
$+ \dfrac{3x^4}{8a^4}\left(1+\dfrac{x}{a}\right) + \dfrac{5x^6}{16a^6}\left(1+\dfrac{x}{a}\right) + \text{\&c.}$

7. $\sqrt{1+x+x^2+x^3+\text{\&c.}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \text{\&c.}$

8. $x^2 = (x+1)(x+2) - 3(x+1) + 1$, and $x^3 = (x+1)(x+2)(x+3) - 6(x+1)(x+2) + 7(x+1) - 1$.

9. $x^3 = x + 3x(x-1) + x(x-1)(x-2)$, and $x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3)$.

10. $4x^2 = 1 + (2x-1)(2x+1)$, and $16x^4 = 1 + 10(2x-1)(2x+1) + (2x-3)(2x-1)(2x+1)(2x+3)$.

11. $(x+a)^2 = (a-1)^2 + (2a-3)(x+1) + (x+1)(x+2)$, and $x(x^2+2)^2 = 9x + 9(x-1)x(x+1) + (x-2)(x-1)x(x+1)(x+2)$.

12. $x^2 = \dfrac{x(x+1)}{1.2} + \dfrac{(x-1)x}{1.2}$, and $x^3 = \dfrac{x(x+1)(x+2)}{1.2.3}$
$+ \dfrac{4(x-1)x(x+1)}{1.2.3} + \dfrac{(x-2)(x-1)x}{1.2.3}.$

13. $\dfrac{1}{(x+1)(x+3)} = \dfrac{1}{(x+1)(x+2)} - \dfrac{1}{(x+1)(x+2)(x+3)}$,

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{1}{(x+1)(x+2)} - \frac{3}{(x+1)(x+2)(x+3)}.$$

14. $\dfrac{2}{x(x+2)} = \dfrac{1}{x} - \dfrac{1}{x+2}$, and $\dfrac{x^2+x+1}{x^3(x^2+1)^2}$
$= \dfrac{1}{x^3} + \dfrac{1}{x^2} - \dfrac{1}{x} - \dfrac{1}{(x^2+1)^2} + \dfrac{x-1}{x^2+1}.$

15. $$\frac{x+c}{(x-a)(x-b)} = \frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)},$$

and $$\frac{x^2}{(x-a)(x-b)} = 1 + \frac{a^2}{(a-b)(x-a)} - \frac{b^2}{(a-b)(x-b)}.$$

16. $$\frac{x^3}{(x-a)(x-b)} = x + a + b + \frac{a^3}{(a-b)(x-a)}$$

$$- \frac{b^3}{(a-b)(x-b)}, \text{ and } \frac{x^5}{(x-a)(x-b)} = x^3 + \frac{a^2-b^2}{a-b}x^2 + \frac{a^3-b^3}{a-b}x$$

$$+ \frac{a^4-b^4}{a-b} + \frac{a^5}{(a-b)(x-a)} - \frac{b^5}{(a-b)(x-b)}.$$

17. $$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(a-b)(a-c)(x-a)}$$

$$+ \frac{1}{(b-a)(b-c)(x-b)} + \frac{1}{(c-a)(c-b)(x-c)}, \text{ and } \frac{2\sqrt{-1}}{1-4x+5x^2}$$

$$= \frac{2+\sqrt{-1}}{\{1-(2+\sqrt{-1})x\}} + \frac{2-\sqrt{-1}}{\{1-(2-\sqrt{-1})x\}}.$$

18. $$\frac{x^2}{(x^2+a^2)(x-b)^2} = \frac{b^2}{(a^2+b^2)(x-b)^2} + \frac{2a^2b}{(a^2+b^2)^2(x-b)}$$

$$- \frac{2a^2bx - a^2(a^2-b^2)}{(a^2+b^2)^2(x^2+a^2)}.$$

19. $$\frac{\alpha x^2+\beta x+\gamma}{(x-a)^3(x+b)} = \frac{\alpha a^2+\beta a+\gamma}{(a+b)(x-a)^3} + \frac{\alpha a(a+2b)+\beta b-\gamma}{(a+b)^2(x-a)^2}$$

$$+ \frac{\alpha b^2-\beta b+\gamma}{(a+b)^3(x-a)} - \frac{\alpha b^2-\beta b+\gamma}{(a+b)^3(x+b)}.$$

20. $$\frac{(a+x)^m}{1.2.3.\&\text{c. } m} = \frac{a^m}{1.2.3.\&\text{c. } m} + \frac{a^{m-1}x}{1.\{1.2.3.\&\text{c. } (m-1)\}}$$

$$+ \frac{a^{m-2}x^2}{1.2.\{1.2.3.\&\text{c. } (m-2)\}} + \&\text{c.}$$

21. $$(a+x)^m = (a^m+x^m) + max(a^{m-2}+x^{m-2}) + \frac{m(m-1)}{1.2}$$

$$a^2x^2(a^{m-4}+x^{m-4}) + \frac{m(m-1)(m-2)}{1.2.3}a^3x^3(a^{m-6}+x^{m-6}) + \&\text{c.}$$

$$+ \frac{m(m-1)(m-2)\ \&\text{c. } (\frac{1}{2}m+1)}{1.2.3.\ \&\text{c. } \frac{1}{2}m} a^{\frac{m}{2}} x^{\frac{m}{2}}, \text{ when } m \text{ is even.}$$

22. $(a+x)^m = (a^m + x^m) + m a x(a^{m-2} + x^{m-2}) + \dfrac{m(m-1)}{1.2}$

$$a^2 x^2(a^{m-4} + x^{m-4}) + \frac{m(m-1)(m-2)}{1.2.3} a^3 x^3(a^{m-6} + x^{m-6}) + \&c.$$

$$+ \frac{m(m-1)(m-2)\ \&c.\ \frac{1}{2}(m+3)}{1.2.3.\&c.\ \frac{1}{2}(m-1)} a^{\frac{m-1}{2}} x^{\frac{m-1}{2}} (a+x),$$

when $m$ is odd.

23. Shew that the $n^{\text{th}}$ and $(n+1)^{\text{th}}$ coefficients of the expansion of $(a+x)^m$ are together equal to the $(n+1)^{\text{th}}$ coefficient of the expansion of $(a+x)^{m+1}$.

24. If $S$ be the sum of the odd terms of the expansion of the binomial $(a+x)^m$, and $s$ the sum of the even terms: then will $S^2 - s^2 = (a^2 - x^2)^m$.

25. Prove that four times the product of the sums of the odd and even terms of the expansion of the binomial $(a+x)^m$ is equivalent to $(a+x)^{2m} - (a-x)^{2m}$.

26. If the coefficients of the expansion of $(a-x)^m$ be multiplied by $a$, $a+b$, $a+2b$, &c. in order, the result $=0$: required a proof.

27. If the coefficients of the expansion of $(a-x)^m$ be multiplied by $1^n$, $2^n$, $3^n$, &c. in order, the result $=0$, if $m$ be greater than $n$: required a proof. N.73

28. If the terms of the expansion of $(a+x)^m$ be multiplied by $0$, $x$, $2x$, $3x$, &c. in order, find the value of the resulting series.

29. If $S$ be the sum of all the coefficients of the expansion of $(a+x)^{2m}$, and $C$ the coefficient of the middle term, it is required to prove that

$$\{1.3.5.\&c.\ (2m-1)\}\, S = \{2.4.6.\&c.\ 2m\}\, C.$$

30. If $A_0$, $A_1$, $A_2$, $A_3$, &c. be the coefficients of the terms of the expansion of $(1+x)^m$, prove that

$$A_0 A_2 + A_1 A_3 + A_2 A_4 + \&c.\ A_{m-2} A_m = \frac{2m(2m-1)\ \&c.\ (m-1)}{1.2.3.\&c.\ (m+2)}.$$

31. In the expansion of $(a+x)^m$, prove that the $(n+2)^{\text{th}}$ term will exceed the $(n+1)^{\text{th}}$ term so long as $(m-n)x$ is greater than $(n+1)\,a$.

32. Given the $n^{\text{th}}$ and $(n+2)^{\text{th}}$ terms of an expanded binomial $(1+x)^m$, to find the index $m$.

33. If $K$ and $L$ be any two consecutive coefficients of the expansion of $(1+v)^m$, prove that the coefficients after $L$, are $L\dfrac{mL-K}{(m+2)K+L}$, $L\dfrac{mL-K}{(m+2)K+L}\,\dfrac{(m-1)L-2K}{(m+3)K+2L}$, &c.

34. $$\sqrt{a+x}=\sqrt{a}\left\{1+\tfrac{1}{2}\frac{x}{a+x}+\frac{1.3}{2.4}\frac{x^2}{(a+x)^2}+\frac{1.3.5}{2.4.6}\frac{x^3}{(a+x)^3}+\&c.\right\}.$$

35. $$\left(\frac{1+x}{1-x}\right)^m=1+m\left(\frac{2x}{1+x}\right)+\frac{m(m+1)}{1.2}\left(\frac{2x}{1+x}\right)^2+\frac{m(m+1)(m+2)}{1.2.3}\left(\frac{2x}{1+x}\right)^3+\&c.$$

36. $$\left(\frac{1+2x}{1+x}\right)^m=1+m\left(\frac{x}{1+2x}\right)+\frac{m(m+1)}{1.2}\left(\frac{x}{1+2x}\right)^2+\frac{m(m+1)(m+2)}{1.2.3}\left(\frac{x}{1+2x}\right)^3+\&c.$$

37. The cube roots of $7+5\sqrt{2}$ and $11\sqrt{2}+3\sqrt{27}$, are $\sqrt{2}+1$ and $\sqrt{2}+\sqrt{3}$ respectively.

38. If $N$ be the $n^{\text{th}}$ term of the expansion of $(1+x)^m$, prove that the series after the first $n$ terms is represented by

$$Nx\left(1-\frac{m+1}{n}\right)+Nx^2\left(1-\frac{m+1}{n}\right)\left(1-\frac{m+1}{n+1}\right)$$
$$+Nx^3\left(1-\frac{m+1}{n}\right)\left(1-\frac{m+1}{n+1}\right)\left(1-\frac{m+1}{n+2}\right)+\&c.$$

39. In the expansion of $(a+b+c+d+\&c.)^5$, find the whole coefficient of $a\,b^2\,c\,d$.

40. Find the coefficient of $x^n$ in the expansion of $(1-x+x^2)^m$, and thence find its developement to six terms.

41. The coefficient of the middle term of the expansion of

$$(1+x+x^2)^m = 1 + \frac{m(m-1)}{1.2} + \frac{m(m-1)(m-2)(m-3)}{(1.2)^2}$$

$$+ \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{(1.2.3)^2} + \&c.$$

$$+ \frac{m(m-1)(m-2)\,\&c.\,(m-2r+1)}{(1.2.3.\&c.\,r)^2} + \&c.$$

42. In the expansion of $a^x$, shew that if any term be less than that which immediately precedes it, the same will hold good of all subsequent terms; and find the greatest term.

## CHAP. VIII.

### MISCELLANEOUS EXAMPLES.

1. PROVE that the ratio of $a+2b : a+b$ is greater than the ratio of $a+3b : a+2b$.

2. Which of the ratios $x^2+y^2 : x+y$ and $x^3+y^3 : x^2+y^2$, is the greater? Answer $x^3+y^3 : x^2+y^2$.

3. Of the ratios $1+x : 1+2x$ and $(1+x)^m : (1+2x)^m$, which is the greater? Answer $1+x : 1+2x$.

4. Express the ratio $4a^3-3a^2x-4ax^2+3x^3 : 3a^3-2a^2x-3ax^2+2x^3$ in its lowest terms. Answer $4a-3x : 3a-2x$.

5. What quantity must be added to each of the terms of the ratio $a : b$, that it may become $c : d$? Answer $\frac{ad-bc}{c-d}$.

6. What is the ratio arising from the composition of the ratios $a+x : b$, $a^2-x^2 : a^2$ and $b : a-x$?

Answer $(a+x)^2 : a^2$.

7. Of the ratios $a^2+b^2 : a^2-b^2$ and $a^2+2ab+b^2 : a^2-2ab+b^2$? Answer $(a^2+b^2)(a+b) : (a-b)^3$.

8. Of $x^2+1 : x^2-1$, $x^4+1 : x^4-1$ and $(x^2-1)^2(x^4-1) : x^8+1$? Answer $x^8-1 : x^8+1$.

9. If a series of quantities increase or decrease by the same common difference, the ratio between any two equidistant terms decreases or increases.

10. If $a : b :: c : d$, it is required to prove that $a \pm mb : b :: c \pm md : d$ and $a : ma \pm b :: c : mc \pm d$.

11. Also, $a^2 - b^2 : ab :: c^2 - d^2 : cd$ and $ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd$.

12. If $a+b : c+d :: c-d : a-b$, then will $a+c : b+d :: b-d : a-c$.

13. If $a : b :: c : d :: e : f$, shew that $a : b :: ma \pm nc \pm qe : mb \pm nd \pm qf$.

14. If $a : b :: c : d :: e : f$, then will $a-e : b-f :: c : d$, and $a+mc-e : b+md-f :: a : b$.

15. If $x$ be to $y$ in the duplicate ratio of $a$ to $b$, and $a$ be to $b$ in the subduplicate ratio of $a+x : a-y$, then will $2x : a :: x-y : y$.

16. A garrison of 1000 men was victualled for 30 days: after 10 days it was reinforced and then the provisions were exhausted in 5 days: required the number of men in the reinforcement. Answer 3000.

17. If 248 men can dig a trench 230 yards long, 3 wide and 2 deep in 5 days of 11 hours each: in how many days of 9 hours each can 24 men dig a trench 420 yards long, 5 wide and 3 deep? Answer $288\frac{59}{207}$.

18. $A$ and $B$ travelled on the same road and at the same rate from $H$ to $L$: at the 50th mile stone from $L$, $A$ overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hours; and 2 hours afterwards met a stage waggon, which was moving at the rate of 9 miles in 4 hours. $B$ overtook the same drove of geese at the 45th mile stone, and met the same stage waggon exactly 40 minutes before he came to the 31st mile stone. Where was $B$ when $A$ reached $L$?

Answer 25 miles from $L$.

19. The arithmetic mean between two numbers exceeds the geometric by 13, and the geometric exceeds the harmonic by 12: what are the numbers? Answer 104 and 234.

20. Of the arithmetic, geometric and harmonic means between two quantities, given any two, to find the quantities.

21. If the arithmetic mean between $a$ and $b$ be twice as great as the geometric mean, prove that $\frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$.

22. If the arithmetic mean between $a$ and $b$ be $m$ times the harmonic, then $\frac{a}{b} = \frac{\sqrt{m}+\sqrt{m-1}}{\sqrt{m}-\sqrt{m-1}}$.

23. If the geometric mean between $a$ and $b$ be $m$ times the harmonic, then $\frac{a}{b} = \frac{m+\sqrt{m^2-1}}{m-\sqrt{m^2-1}}$.

24. If $a$ and $b$ be nearly equal to each other, prove that the arithmetic, geometric and harmonic means between them are nearly equal.

25. If $y$ be the harmonic mean between $x$ and $z$, and $x$ and $z$ be respectively the arithmetic and geometric mean between $a$ and $b$, then $\frac{y}{z} = \frac{a+b}{\left(\sqrt[4]{\frac{a}{b}}+\sqrt[4]{\frac{b}{a}}\right)^2}$.

26. If $\sqrt[n]{x}+m\sqrt[n]{y} : \sqrt[n]{x}-m\sqrt[n]{y} :: \sqrt[2n]{x}+m\sqrt[2n]{x-y} : \sqrt[2n]{x}-m\sqrt[2n]{x-y}$, then will $\left(\frac{x}{y}\right)^2 = \frac{\sqrt{5}+1}{\sqrt{5}-1}$.

27. If $A \propto B$ and $B \propto C$, then will $A \propto mB \pm nC$.

28. If $A \propto B$ and when $A=m$, $B=n$; then $nA=mB$.

29. If $a+b \propto a-b$, prove that $a^2+b^2 \propto ab$, and $a^3+b^3 \propto ab(a+b)$ or $\propto ab(a-b)$.

30. If $z$ consist of two parts one of which $\propto x$ and the other $\propto y$; also when $x=y$, then $z=2mx$, and when $x=-y$, then $z=2nx$: express $z$ in terms of $x$ and $y$.

31. Compare the ratio $m^2+n^2 : 2mn$ with the ratio $m : n$, both when $m$ is greater and less than $n$.

32. Prove that the ratio $a\left(1-\frac{2c}{a}\right) : \left(1-\frac{2cx}{a}\right)^{\frac{3}{2}}$ is nearly $a-2c+3cx : 1$, if $\frac{c}{a}$ be very small compared to unity.

33. The ratio $a^2(1-x)+b^2 : x^3$ is nearly equal to the ratio $(a^2+b^2)(1-x)^{\frac{a^2}{a^2+b^2}} : x^3$, when $x$ is small compared to 1.

34. Shew that $\dfrac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}}{1+x+\sqrt{1+x}} : 1$ is nearly equal to $1-\frac{5}{6}x : 1$ or $1 : 1+\frac{5}{6}x$, when $x$ is very small compared to 1. What is the value of the ratio when $x$ is very large?

35. Compare the ratios $\sqrt{2}-1 : \sqrt{2}$ and $\sqrt{3}-1 : \sqrt{3}$; also, $\sqrt{7} : 2\sqrt[4]{3}$ and $\sqrt{6}-\sqrt{5} : \sqrt{8}-\sqrt{7}$.

## CHAP. IX.

### MISCELLANEOUS EXAMPLES.

1. The $10^{\text{th}}$ term of $2+5+8+$ &c. is 29, and the sum of 10 terms is 155.

2. The $13^{\text{th}}$ term of $3+9+15+$ &c. is 75, and the sum of 13 terms is 507.

3. The $100^{\text{th}}$ term of $1+9+17+$ &c. is 793, and the sum of 100 terms is 39700.

4. The $24^{\text{th}}$ term of $7+5+3+$ &c. is $-39$, and the sum of 24 terms is $-384$.

5. The $20^{\text{th}}$ term of $4-3-10-17-$ &c. is $-129$, and the sum of 20 terms is $-1250$.

6. The $12^{\text{th}}$ term of $1+\frac{3}{2}+2+\frac{5}{2}+$ &c. is $6\frac{1}{2}$, and the sum of 12 terms is 45.

7. The $16^{\text{th}}$ term of $15+\frac{44}{3}+\frac{43}{3}+$ &c. is 10, and the sum of 16 terms is 200.

8. The $6^{th}$ term of $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+$ &c. is $-\frac{1}{3}$, and the sum of 6 terms is 1.

9. The $n^{th}$ term of $\frac{1}{3}+\frac{5}{6}+\frac{4}{3}+$ &c. is $\frac{1}{2}n-\frac{1}{6}$, and the sum of $n$ terms is $\frac{1}{12}n+\frac{1}{4}n^2$.

10. The $n^{th}$ term of $\frac{1}{2}+\frac{1}{6}-\frac{1}{6}-$ &c. is $\frac{5}{6}-\frac{1}{3}n$, and the sum of $n$ terms is $\frac{2}{3}n-\frac{1}{6}n^2$.

11. The $n^{th}$ term of $\dfrac{n-1}{n}+\dfrac{n-2}{n}+\dfrac{n-3}{n}+$ &c. is 0, and the sum of $n$ terms is $\frac{1}{2}(n-1)$.

12. The first term is $n^2-n+1$ and the common difference is 2: prove that the sum of $n$ terms is $n^3$, and thence shew that $1^3=1$, $2^3=3+5$, $3^3=7+9+11$, $4^3=13+15+17+19$, &c.

13. Prove that $\dfrac{a-b}{a+b}+\dfrac{3a-2b}{a+b}+$ &c. to $n$ terms $=\dfrac{n}{a+b}\left\{na-\frac{1}{2}(n+1)b\right\}$.

14. The sum of the first two terms of an arithmetical progression is 4, and the $5^{th}$ term is 9: find the series.

Answer 1, 3, 5, 7, 9, &c.

15. The first term of an arithmetical progression is $a$, and the $n^{th}$ term is equal to $m$ times the common difference: find the series. Answer $a$, $\dfrac{m-n+2}{m-n+1}a$, $\dfrac{m-n+3}{m-n+1}a$, &c.

16. The first two terms of an arithmetical progression being together $=18$ and the next three terms $=12$: how many terms must be taken to make 28? Answer 4.

17. The sum of an arithmetical series is 1455, the first term 5 and the number of terms 30: what is the common difference? Answer 3.

18. The latter half of $2n$ terms of any arithmetical series is equal to half the sum of $3n$ terms of the same series.

19. The difference between the sums of $m$ and $n$ terms of an arithmetical progression : the sum of $m+n$ terms :: $m-n : m+n$.

20. If $S_1$, $S_2$ and $S_3$ be the sums of $n$ terms of three arithmetical progressions, of which 1 is the first term and the respective common differences 1, 2 and 3: then $S_1+S_3=2S_2$.

21. The sum of $n$ terms of an increasing arithmetical progression, whose common difference is equal to the least term, is the sum of $n+1$ magnitudes, each of which is half the greatest term.

22. The sum of an even number of terms of an arithmetical progression, whose common difference is equal to the least term, will be four times the sum of half that number of terms diminished by half the last term.

23. There are $p$ arithmetic progressions, each beginning from 1, and the common differences are 1, 2, 3, &c. $p$: shew that the sum of their $n^{\text{th}}$ terms $=\frac{1}{2}\{(n-1)p^2+(n+1)p\}$.

24. $S_1$, $S_2$, $S_3$, &c. $S_p$ are the sums of $p$ arithmetical progressions, each continued to $n$ terms: the first terms are 1, 2, 3, &c. and the common differences 1, 3, 5, &c.: then is $S_1+S_2+S_3+\text{\&c.}+S_p=\frac{1}{2}np(np+1)$.

25. Prove that 1, 3, 5, 7, &c. is the only arithmetical progression beginning with 1, in which the sum of the first half of any even number of terms has to the sum of the second half the same constant ratio; and find that ratio.

26. Insert four arithmetic means between 193 and 443, and three between 117 and 477.

Answer 243, 293, 343, 393 and 207, 297, 387.

27. Find four arithmetic means between 2 and $-18$, and five between $\frac{1}{2}$ and $-\frac{1}{2}$.

Answer $-2$, $-6$, $-10$, $-14$ and $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$.

28. The sum of $m$ arithmetic means between 1 and 19 : the sum of the first $m-2$ of them :: 5 : 3; find the number.

Answer 8.

29. There are $m$ arithmetic means between 1 and 31, and the $7^{\text{th}}$ : $(m-1)^{\text{th}}$ :: 5 : 9; find the number of means.

Answer 14.

30. In an arithmetical progression, if the $(p+q)^{\text{th}}$ term $=m$, and the $(p-q)^{\text{th}}$ term $=n$: then the $p^{\text{th}}$ term $=\frac{1}{2}(m+n)$, and the $q^{\text{th}}$ term $=m-(m-n)\dfrac{p}{2q}$.

31. The sum of $n$ terms of the series $3.5+5.7+7.9+$ &c. is $\frac{1}{3}(4n^3+18n^2+23n)$.

32. The sum of $n$ terms of $1.2.3+2.3.4+3.4.5+$ &c. is $\frac{1}{4}\{n(n+1)(n+2)(n+3)\}$.

33. The sum of $n$ terms of $1^2.2+2^2.3+3^2.4+$ &c. is $\frac{1}{12}\{n(n+1)(3n^2+7n+2)\}$.

34. Prove that $A_1.A_2+A_2.A_3+\text{&c.}+A_{n-1}.A_n$ may always be expressed in finite terms, when $A_1$, $A_2$, $A_3$, &c. are in arithmetical progression.

35. If $S_n$, $S_{n+1}$, $S_{n+2}$, &c. denote the sums of $n$, $n+1$, $n+2$, &c. terms of an arithmetical progression whose first term is $a$ and common difference $d$, prove that $S_n+S_{n+1}+S_{n+2}+$ &c. to $n$ terms $=n(3n-1)\dfrac{a}{1.2}+n(n-1)(7n-2)\dfrac{d}{1.2.3}$.

36. The $n^{\text{th}}$ term of $1+3+9+27+$ &c. is $3^{n-1}$, and the sum of $n$ terms is $\frac{1}{2}(3^n-1)$.

37. The $n^{\text{th}}$ term of $1-2+2^2-2^3+$ &c. is $\pm 2^{n-1}$, and the sum of $n$ terms is $\frac{1}{3}(1\mp 2^n)$.

38. The $n^{\text{th}}$ term of $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+$ &c. is $\dfrac{3^{n-2}}{2^{n-1}}$, and the sum of $n$ terms is $\frac{1}{3}\left(\dfrac{3^n-2^n}{2^{n-1}}\right)$.

39. The $n^{\text{th}}$ term of $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$ &c. is $(\frac{2}{3})^{n-4}$, and the sum of $n$ terms is $\frac{1}{8}\left(\dfrac{3^n-2^n}{3^{n-4}}\right)$.

40. The $n^{\text{th}}$ term of $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-$ &c. is $\pm\dfrac{2^{n-1}}{5.3^{n-1}}$, and the sum of $n$ terms is $\frac{1}{25}\left(\dfrac{3^n\mp 2^n}{3^{n-1}}\right)$.

41. The $n^{\text{th}}$ term of $\dfrac{1}{\sqrt{2}} + \frac{1}{2} + \dfrac{1}{2\sqrt{2}} + \text{\&c.}$ is $\dfrac{1}{\sqrt{2^n}}$, and the sum of $n$ terms is $\dfrac{1}{\sqrt{2^n}}\left(\dfrac{\sqrt{2^n}-1}{\sqrt{2}-1}\right)$.

42. The $n^{\text{th}}$ term of $\frac{2}{5} - \sqrt{\frac{2}{5}} + 1 - \text{\&c.}$ is $\pm\left(\frac{2}{5}\right)^{\frac{1}{2}(n+1)}$, and the sum of $n$ terms is $\dfrac{2}{5^{\frac{1}{2}(n+1)}}\left\{\dfrac{5^{\frac{1}{2}n} \mp 2^{\frac{1}{2}n}}{5^{\frac{1}{2}} + 2^{\frac{1}{2}}}\right\}$.

43. The sum of $100 + 40 + 16 + \text{\&c.}$ *in infinitum* is $166\frac{2}{3}$.

44. The sum of $1 + \frac{2}{3} + \frac{4}{9} + \text{\&c.}$ *in infinitum* is 3.

45. The sum of $\frac{1}{6} - \frac{2}{9} + \frac{8}{27} - \text{\&c.}$ *in infinitum* is $\frac{1}{14}$.

46. Of $x^{\frac{5}{2}} - bx + \dfrac{b^2}{\sqrt{x}} - \text{\&c.}$ *in infinitum* is $\dfrac{x^4}{x^{\frac{3}{2}} + b}$.

47. Of $\frac{1}{3} + \dfrac{1}{6\sqrt{-1}} - \frac{1}{12} - \text{\&c.}$ *in inf.* is $\dfrac{2\sqrt{-1}}{6\sqrt{-1}-3}$.

48. If $y = \sqrt{a} - x + \dfrac{x^2}{\sqrt{a}} - \dfrac{x^3}{a} + \dfrac{x^4}{a\sqrt{a}} - \text{\&c.}$, it is required to prove that $\dfrac{1}{x} = \dfrac{y}{a} + \dfrac{y^2}{a\sqrt{a}} + \dfrac{y^3}{a^2} + \dfrac{y^4}{a^2\sqrt{a}} + \text{\&c.}$

49. Given the sum, and the sum of the squares of the terms of a geometrical progression, to exhibit it.

50. In any geometrical progression, the sums of every $n$ successive terms are in geometrical progression: find the sum of $m$ such sums, and shew that the result is the same as that for the sum of $mn$ terms of the first series.

51. In every geometrical progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

52. If $s_1$ and $s_2$ denote the sums of $n$ and $2n$ terms of a geometrical progression: express the first term and the common ratio in terms of $s_1$, $s_2$ and $n$.

53. In an ascending geometric series, the sum of the first and last terms is always greater than the sum of any other two terms equally distant from the extremes.

54. In any geometrical progression of an even number of terms, prove that the sum of the odd terms is to the sum of the even terms as $1 : r$.

55. The sum of a series of quantities in geometrical progression, is equal to the sum of all the terms except the last multiplied by the common ratio.

56. If $S$ be the sum of $n$ terms of a geometrical progression whose first term is $s_1$, common ratio $r$ and number of terms $n$: also, if $s_2$, $s_3$, $s_4$, &c. be the sums of the first two, three, four, &c. terms: then will

$$(S+s_1)+(S+s_2)+(S+s_3)+\&c.$$
$$=\frac{s_1}{r-1}\left\{n(r^n-2)+\frac{r}{r-1}(r^n-1)\right\}.$$

57. If $S_1$, $S_2$, $S_3$, &c. $S_n$ be the sums of $n$ geometrical progressions whose first terms are $a$, $2a$, $3a$, &c. $na$; $n$ the number of terms, and $r$ the common ratio in each: then will

$$S_1+S_2+S_3+\&c.+S_n=\frac{n(n+1)}{2}\left(\frac{r^n-1}{r-1}\right)a.$$

58. Insert three geometric means between 39 and 3159, and also three between 37 and 2997.

Answer 117, 351 and 1053: 111, 333 and 999.

59. Find three geometric means between $\frac{1}{3}$ and $\frac{1}{48}$, and four between $\frac{1}{2}$ and 512. Answer $\frac{1}{6}$, $\frac{1}{12}$ and $\frac{1}{24}$: 2, 8, 32 and 128.

60. In a geometrical progression, if the $(p+q)^{\text{th}}$ term $=m$, and the $(p-q)^{\text{th}}=n$, then will the $p^{\text{th}}$ term $=\sqrt{mn}$, and the $q^{\text{th}}$ term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$.

61. If $a$ and $l$ be the first and $n^{\text{th}}$ terms of a decreasing geometrical progression, it is required to prove that

$$\Sigma=\cancel{\frac{a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}-l^{\frac{1}{n-1}}}}. \qquad =\frac{a^{\frac{n}{n-1}}}{a^{\frac{1}{n-1}}-l^{\frac{1}{n-1}}}$$

62. If $S_1$ represent the sum of an infinite geometrical progression whose first term is $a$ and ratio $r$, $S_2$ the sum of the squares, $S_3$ the sum of the cubes, &c. of the terms: then will

$$\frac{1}{S_1} \pm \frac{1}{S_2} + \frac{1}{S_3} \pm \text{\&c. } in\ infinitum = \frac{1}{a \mp 1} - \frac{r}{a \mp r}.$$

63. If $S_1$, $S_2$, $S_3$, &c. $S_n$ be the sums of $n$ geometrical progressions continued *in infinitum*, the first term of each being 1 and the common ratios $\frac{1}{r^1}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c. $\frac{1}{r^n}$: prove that

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \text{\&c.} + \frac{1}{S_n} = n - \frac{r^n - 1}{r^n (r-1)}.$$

64. Let $S_1$, $S_2$, $S_3$, &c. denote the sums of an infinite number of infinite decreasing geometrical progressions whose first terms are $a$, $a^2$, $a^3$, &c., and common ratios $r$, $2r$, $3r$, &c.:

$$\text{then } \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \text{\&c.} = \frac{a(1-r)-1}{(a-1)^2}.$$

65. If $S = (x-y) + \left(\frac{y^2}{x} - \frac{y^3}{x^2}\right) + \text{\&c.}$ to $n$ terms, and $\Sigma$ denote the sum of the same series *in infinitum*: then will

$$S : \Sigma :: x^{2n} - y^{2n} : x^{2n}.$$

66. The sums of two infinite geometrical series are $S$ and $S'$, so that $\frac{S}{S'} = \frac{4}{9}$: of the former the first two terms are 40 and 35, and of the latter the second term is $46\frac{19}{36}$: find both the series. Answer $a = 40$, $r = \frac{7}{8}$ and $a' = 50$, $r' = \frac{67}{72}$.

67. Insert four harmonic means between 2 and 12.

Answer $2\frac{2}{5}$, 3, 4 and 6.

68. The sum of three terms of an harmonic series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$: find the series. Answer $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

69. There are four numbers, the first three of which are in arithmetical progression, and the last three in harmonical: it is required to prove that the products of the extremes and means are equal.

70. If $S$ and $S'$ be the sums of two infinite series whose first terms are 1 and common ratios $r$ and $r'$: then are $S$, $S'$, $r$ and $r'$ in harmonical progression.

71. Continue both ways the harmonical progression 2, 3, 6. The series is &c. $1\frac{1}{5}$, $1\frac{1}{2}$, 2, 3, 6, $\infty$, $-6$, $-3$, $-2$, &c.

72. If quantities be in geometrical progression, their differences are in the same geometrical progression.

73. If $a$, $b$, $c$, $d$, &c. be in geometrical progression; then will $\frac{1}{a^2-b^2}$, $\frac{1}{b^2-c^2}$, $\frac{1}{c^2-d^2}$, &c. be in geometrical progression. Find also their sum.

74. If $a$, $b$ and $c$ be in geometrical progression, then $a^2+b^2+c^2=(a+b+c)(a-b+c)$, and therefore $>(a-b+c)^2$.

75. Also, $a^2+b^2+c^2-(a-b+c)^2=2b(a-b+c)$, $3(a^2+b^2+c^2)-(a+b+c)^2=2(a+b+c)(a-2b+c)$ and $3(a-b+c)^2-(a^2+b^2+c^2)=2(a-b+c)(a-2b+c)$.

76. If $a$, $b$, $c$ and $d$ be in geometrical progression, then will $(a+3b+3c+d)bc=(b+c)^3$, $4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2=(a-b)^2+(c-d)^2+2(a-d)^2$ and $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$.

77. Also, $2(a^2+b^2+c^2+d^2)-(a+b+c+d)^2=(a-b)^2+(c-d)^2-2(b+c)^2$ and $a^2+b^2+c^2+d^2+(a+b+c+d)(b+c) : (a+b+c+d)^2-(a^2+b^2+c^2+d^2) :: a+b+c+d : 2(b+c)$.

78. If $a$, $b$, $c$, $d$ and $e$ be in geometrical progression, then is $2(a+c+e)>3(b+d)$, $c(a+2c+e)=(b+d)^2$ and $(a+3b+3c+d)(b-c)=(a-b-c+d)(b+c)^2$.

79. The $n^{\text{th}}$ term of $2+6+14+30+$ &c. is $2^{n+1}-2$, and the sum of $n$ terms is $2^{n+2}-(2n+4)$.

80. The $n^{\text{th}}$ term of $4+10+28+82+$ &c. is $3^n+1$, and the sum of $n$ terms is $\frac{1}{2}3^{n+1}+n-\frac{3}{2}$.

81. The $n^{\text{th}}$ term of $3+6+11+20+$ &c. is $2^n+n$, and the sum of $n$ terms is $2^{n+1}+\frac{1}{2}(n^2+n-4)$.

82. Compare the values of the respective infinite series $1+2x+3x^2+4x^3+$ &c. and $1-2x+3x^2-4x^3+$ &c.

Answer $(1+x)^2 : (1-x)^2$.

## CHAP. X.

### MISCELLANEOUS EXAMPLES.

1. The number of variations two together : the number three together :: 1 : 5; required the number of things.

Answer 7.

2. The number of things : the number of variations three together :: 1 : 20; what is that number? Answer 6.

3. The number of combinations of $m$ things taken four together : the number taken two together :: 15 : 2; find the value of $m$. Answer 12.

4. Find the number of different triangles into which a polygon of $m$ sides may be divided by joining the angular points. Answer $\frac{1}{6}m(m-1)(m-2)$.

5. There is a certain number of things of which the variations taken eight together is 80, and taken ten together is 960: how many must be taken away from the original number, that of the remaining things taken two together the combinations may be 15? Answer 6.

6. If the number of variations of a certain number of things taken three together be six times as great as the number of combinations of the same things taken four together: how many things are there? Answer 7.

7. How many days can five persons be placed in different positions about a table at dinner? Answer 120.

8. How many different numbers can be made out of the figures 1220005555? Answer 12600.

9. How many different numbers can be formed by taking five digits out of the ten which compose the common scale of notation? Answer 30240.

10. How many combinations can be made of two letters out of the twenty-six letters of the alphabet? Answer 325.

11. Find the whole number of permutations and combinations that can be made out of the four letters $a$, $b$, $c$, $d$ when they are taken by two's, three's and by fours. Answer 340.

12. How many words can be made with five letters of the alphabet, admitting that a number of consonants alone will not make a word? Answer 5100480.

13. How many different numbers can be made out of 1 unit, 2 twos, 3 threes, 4 fours and 5 fives, taken 5 at a time?

Answer 2111.

14. How many changes are there in throwing five dice?

Answer 7776.

15. How many changes are there contained in the permutations of $abcdefg$ which begin with one of the letters?

Answer 720.

16. How many do they contain beginning with $ab$, with $abc$ and with $abcd$? Answer 120, 24 and 6.

17. How many deals may a person play at the game of whist without having the same hand twice?

Answer 635013559600.

18. The number of variations of $m$ things taken $r$ together : the number taken $r-1$ together :: 10 : 1, and the corresponding numbers of combinations are as 5 : 3; find the values of $m$ and $r$. Answer $m=15$ and $r=6$.

19. If $p_2, p_3$, &c. $p_m$ represent the numbers of permutations that can be formed out of $m$ things taken 2, 3, &c. $m$ together respectively, and $P = p_2\, p_3$ &c. $p_m$, then will

$$P = p_3 p_{m-1}\{(p_3 - p_2)(p_4 - p_3)(p_5 - p_4) \text{ \&c. } (p_{m-1} - p_{m-2})\}.$$

# CHAP. XI.

## MISCELLANEOUS EXAMPLES.

1. Transform 30420 from the quinary to the denary scale of notation. Answer 1985.

2. Express 4567 in a system of notation whose base is 12; and what number in the denary scale is equal to 5432 in the senary? Answer 2787 and 1244.

3. Transform 1828 from the denary scale to the scales whose local values are 11 and 13. Answer 1412 and $tt8$.

4. Transform 1756 and 345 from the common to the duodenary scale, and then find their product.

Answer 1024, 249 and 252710.

5. What is the quotient of 14332216 by 6541 in a scale of which the radix is 7? Answer 1456.

6. Find the quotient of $29t96580$ by $2tt9$ in the duodenary scale. Answer $u7t8$.

7. What is the square root of 13233010 in a system of notation whose base is 4? Answer 2302.

8. Transform 4321 from the quinary to the senary scale of notation. Answer 2414.

9. The number 2577 expressed in a particular scale of notation is 40302: find the radix. Answer 5.

10. What is the radix of the scale of notation wherein a number which is double of 145 will be expressed by the same digits? Answer 15.

11. Determine the weights which must be selected out of the series 1, 2, 4, 8, &c. pounds in order to weigh 159 pounds.

Answer 1, 2, 4, 8, 16, 128.

12. What number of the weights of 1, 3, $3^2$, $3^3$, &c. pounds must be selected to weigh 304 pounds?

Answer $3^5$, $3^4$, $3^2$, 1 and $3^3$, 3.

13. Any number consisting of three figures is divisible by 7, if the first and third figures be the same, and the sum of the first and second a multiple of seven.

14. Any number of four places is divisible by 7, if the first and last digits be the same, and the digit in the place of hundreds be double that in the place of tens.

15. If the number expressed by the last $n$ digits of a number be divisible by $2^n$, the number itself is divisible by $2^n$.

16. In any system of notation whose local value is $a$, in any multiple of $a-1$, the sum of the digits is either equal to $a-1$ or to some multiple of it.

17. Every number in which the digit recurs an even number of times, is divisible by 11.

18. Any number consisting of an even number of places, in a system whose radix is $r$, is divisible by $r+1$, if the corresponding digits from each end be the same.

19. The square of any number of digits less than ten, each of which is unity, will when reckoned from either end, form the same arithmetic series whose common difference is unity and greatest term the number of digits in the root.

20. If any number a multiple of 11 and a number consisting of the same digits in an inverted order be each divided by 11, the sums of the digits in the two quotients are equal.

21. If the sum of the odd digits in a number be $11m+e$ and of the even $11n+e$, this number being divided successively by 11 and 9 leaves the same remainder as $m-n+e$ when divided by 9.

22. What is the value of .71333 &c. *in infinitum?*

Answer $\frac{107}{150}$.

23. Required the value of $.3p\ 3p\ 3p$ &c. *in infinitum* where $p$ contains $\frac{1}{2}(m-1)$ digits. Answer $\dfrac{3p}{10^{\frac{1}{2}(m+1)}-1}$.

24. Find the value of .2534534 &c. *in infinitum* when the radix is 6. Answer $\frac{1244}{2555}$.

25. What is the product of 34.021 by 1.23 when the radix is 5? Answer 104.00133.

26. The length and breadth of a floor are 27 feet 10 inches and 14 feet 6 inches: what is its area?

Answer 403 feet 7 inches.

27. Find the quotient of 1532 feet $9\frac{3}{4}$ inches by 81 feet 9 inches. Answer 18 feet 9 inches.

28. The area of a rectangle is 29 feet 4 parts and its breadth is 2 feet 3 inches 6 parts: what is its length?

Answer 12 feet 8 inches.

29. The length and breadth of the base of a rectangular parallelopiped are 9 feet 6 inches and 4 feet 7 inches: what is its height when the solid content is 152 feet 4 inches 9 parts?

Answer 3 feet 6 inches.

30. The content of a cube is 14 feet 1 inch 4 parts 5 thirds: what is the length of its side? Answer 2 feet 5 inches.

31. There is a number consisting of three digits in geometrical progression; the number is to the sum of its digits :: 124 : 7; and if 594 be added to it, the digits will be inverted: what is it? Answer 248.

32. Prove that the sum of all the numbers of $n$ places which can be formed with the $n$ digits $a$, $b$, $c$, &c. : the sum of all the numbers of $n$ places which can be formed with the $n$ digits $p$, $q$, $r$, &c. of the same scale :: $a+b+c+$ &c. : $p+q+r+$ &c.

## CHAP. XII.

### MISCELLANEOUS EXAMPLES.

1. If $n$ be a whole number, prove that $n^3+5n$ is divisible by 6.

2. If $n$ be any whole number, then will $n(n^2-1)(n^2-4)$ be divisible by 120.

3. If $n$ be any odd number, then $n^5 - n$ will be a multiple of 12.

4. If $n$ be an even number, then will $n(n^2 + 20)$ be divisible by 48.

5. The square of every odd number diminished by 1 is divisible by 8.

6. If from the cube of any even number be subtracted four times the number itself, the remainder will be divisible by 48.

7. If $n$ be greater than 3, prove that the cube root of $n$ will be greater than the fourth root of $n + 1$.

8. The square of every number, except 3 and its multiples, is of the form $3m + 1$.

9. If $a^3 - b^3$ be divisible by 3, then will $(a \pm k)^3 - (b \pm k)^3$ be also divisible by 3.

10. If when any numbers $a_1$, $a_2$, $a_3$, &c. are divided by $n$, the remainders be $r_1$, $r_2$, $r_3$, &c. respectively: then will $r_1\, r_2\, r_3$ &c. be the remainder in the division of $a_1\, a_2\, a_3$ &c. by $n$.

11. If an odd and even square number be added together and the sum be also a square number, the even square is a multiple of 16.

12. If $s$ be the sum of any two numbers, $p$ their product and $q$ their quotient, then will

$$p = s^2 \{q - 2q^2 + 3q^3 - 4q^4 + \text{\&c. } \textit{in infinitum}\}.$$

13. Supposing the sum of 51 cards in a common pack to be $10m + a$, prove the value of the last card to be $10 - a$, the court cards reckoning for 10, and the ace, deuce &c. for 1, 2, &c. respectively: find also the value of $m$.

14. The square of every prime number greater than 2, 3 diminished by 1, is divisible by 24.

15. If 5 be subtracted from the sum of any two consecutive numbers each prime to 3, the remainder will be divisible by 36.

16. The difference of the squares of any two prime numbers, of which the greater exceeds 5, is divisible by 24.

17. The sum of any number of prime numbers in arithmetical progression is a composite number.

18. If we divide $a$, $a^2$, $a^3$, &c. by a prime number $p$, we shall obtain a remainder 1 before we have taken $p$ terms: also, after this remainder, the remainders recur.

19. If $a$ be a prime number and $b$ any other number prime to $a$, shew that if $b^2$, $(2b)^2$, $(3b)^2$, &c. $\{\frac{1}{2}(a-1)b\}^2$ be divided by $a$, they will each leave a different positive remainder.

20. If $m$ be a prime number, and $a$ and $b$ be integers not divisible by $m$, then will $a^{m-1} - b^{m-1}$ be a multiple of $m$.

21. Every prime number of the form $4m+1$ is the sum of two squares.

22. Find the number of divisors of 2160, and also their sum. Answer 40 and 7440.

23. What number multiplied by 48 will make it a complete fourth power? Answer 27.

24. If $a$ and $b$ be prime numbers, the number of numbers prime to $ab$ and less than $ab$ is equal to $(a-1)(b-1)$, unity being considered one of them.

25. Prove that the sum of any two consecutive triangular numbers is a square number.

26. Shew that the ratio between a triangular and square number of the same root approaches to 1 : 2, as that root is increased.

27. Prove that the sum of $n$ terms of the series $1^3 + 3^3 + 5^3 + 7^3 +$ &c. is an hexagonal number whose root is $n^2$.